# The Spherical Table Problem for Odd Functions

Yukina Shirakaga*†

**Abstract**

We extend Naseri Sadr's table theorem from continuous real-valued functions to continuous sections of real line bundles with arbitrary connections over closed connected Riemannian surfaces of odd Euler characteristic. The proof computes the top Stiefel–Whitney class of a rank-three real vector bundle over a closed three-manifold. As applications, we obtain the spherical table theorem for odd functions and give an alternative proof of the known theorem for even functions.

## 1 Introduction

Let $Q = (q_0, q_1, q_2, q_3)$ be the vertices, in cyclic order, of a fixed nondegenerate Euclidean square inscribed in the unit sphere $S^2 \subset \mathbb{R}^3$. The spherical *table problem* asks whether, for every continuous function $h : S^2 \to \mathbb{R}$, there is a rotation $\rho \in \mathrm{SO}(3)$ such that

$$h(\rho q_0) = h(\rho q_1) = h(\rho q_2) = h(\rho q_3). \tag{1}$$

In particular, the size of the square is prescribed. Matschke explicitly formulated the smooth version as a conjecture [5, Conjecture 13]; the continuous version appears in the $K3$ problem list [1, Problem 2.14(d)]. For each fixed square, the two formulations are equivalent: uniform approximation by smooth functions, followed by compactness of $\mathrm{SO}(3)$, gives the continuous statement from the smooth one.

Soberón recently disproved the conjecture [8, Theorem 1]. His counterexample is the restriction to $S^2$ of the cubic polynomial

$$H(x, y, z) = 14z^3 - 6z + y + xz$$

together with the square whose vertex set is

$$Q_* = \left\{ \frac{1}{\sqrt{7}}(\varepsilon\sqrt{3}, \delta\sqrt{3}, 1) : \varepsilon, \delta \in \{\pm 1\} \right\}.$$

There is no rotation of $Q_*$ on whose vertices $H$ takes a common value. Notice that $H$ is neither even nor odd under the antipodal map.

Several important special cases have affirmative answers. When $Q$ lies on a great circle, the conclusion is Dyson's theorem [3]. Alternative proofs were given by Crabb and Jaworowski [2], using stable cohomotopy Euler classes, and by Vassiliev [9], using Stiefel–Whitney classes. For even functions, the conclusion follows from Hausel, Makai, and Szűcs [4, Theorem 10]. Naseri Sadr [7, Theorem 1.1 and Corollary 1.2] established a

*Email: yshirakaga[at]atamath.com

†**Yukina Shirakaga** is a fictional character and a research persona of ATAMATH Project (https://atamath.com/), an experiment in mathematical research and exposition using fictional characters.

general table theorem for Riemannian surfaces of odd Euler characteristic and also deduced the even case on $S^2$.

The purpose of this note is to prove the spherical table theorem for odd functions and to treat the odd and even cases within a common framework. If $A : S^2 \to S^2$ is the antipodal map, the conclusion holds for every prescribed nondegenerate square whenever

$$h \circ A = h \qquad \text{or} \qquad h \circ A = -h.$$

Indeed, an even function descends through $S^2 \to \mathbb{RP}^2$ to an ordinary real-valued function, whereas an odd function descends to a section of the real line bundle associated to the antipodal double cover. This observation leads to a common theorem for sections of real line bundles with connections.

To state the main result, let $(N, g)$ be a closed connected Riemannian surface and fix $a > 0$. An orthonormal frame $(e_1, e_2)$ at $x \in N$ determines four points

$$\exp_x(ae_1), \qquad \exp_x(ae_2), \qquad \exp_x(-ae_1), \qquad \exp_x(-ae_2).$$

Let $\mathcal{L} \to N$ be a real line bundle equipped with a connection. For a path $\gamma$ from $x$ to $y$, write

$$\mathrm{PT}_\gamma : \mathcal{L}_x \longrightarrow \mathcal{L}_y$$

for parallel transport. Given a continuous section $s$ of $\mathcal{L}$, transport its values at the four exponential vertices back along their radial geodesics to $\mathcal{L}_x$. We call the frame an *$\mathcal{L}$-table of radius $a$* if the four resulting elements of $\mathcal{L}_x$ agree; see Definition 2.1.

**Theorem 1.1.** *Let $(N, g)$ be a closed connected Riemannian surface with $\chi(N)$ odd, let $\mathcal{L} \to N$ be a real line bundle equipped with a connection, and let $s$ be a continuous section of $\mathcal{L}$. Then, for every $a > 0$, the section $s$ admits an $\mathcal{L}$-table of radius $a$.*

The connection enters the definition of an $\mathcal{L}$-table and the construction of the test section, but the obstruction bundle depends only on the underlying line bundle. No flatness assumption is imposed. For the trivial line bundle with its trivial connection, Theorem 1.1 recovers Naseri Sadr's table theorem [7, Theorem 1.1], with his diameter parameter $d = 2a$. It also gives an alternative proof: Naseri Sadr's proof uses the geometry of inscribed squares in families of star-shaped plane curves, whereas the argument here computes the relevant characteristic class directly.

A double cover $q : \widetilde{N} \to N$ determines the sign line bundle

$$\mathcal{L}_q = \widetilde{N} \times_{\mathbb{Z}/2} \mathbb{R}_{\mathrm{sgn}} \longrightarrow N$$

with its canonical flat connection. Its continuous sections correspond to continuous functions $h : \widetilde{N} \to \mathbb{R}$ satisfying $h \circ \tau = -h$, where $\tau$ is the deck involution. We obtain the following consequence. Here a table on a Riemannian surface means an exponential table, as defined in Remark 2.2.

**Corollary 1.2.** *Let $(\widetilde{N}, \widetilde{g})$ be a closed connected Riemannian surface with a fixed-point-free isometric involution $\tau$, and put $N = \widetilde{N}/\tau$. Suppose that $\chi(N)$ is odd. If a continuous function $h : \widetilde{N} \to \mathbb{R}$ satisfies either*

$$h \circ \tau = h \qquad \text{or} \qquad h \circ \tau = -h,$$

*then $h$ has an exponential table of every radius $a > 0$.*

*Remark* 1.3. A closed surface of odd Euler characteristic is necessarily nonorientable. Since $\chi(\widetilde{N}) = 2\chi(N)$ for a double cover, the Euler characteristic hypothesis in Corollary 1.2 is equivalent to $\chi(\widetilde{N}) \equiv 2 \pmod 4$.

**Corollary 1.4.** *Equip $S^2$ with its standard round metric, and let $Q = (q_0, q_1, q_2, q_3)$ be the vertices of any nondegenerate square inscribed in $S^2$. If $h : S^2 \to \mathbb{R}$ is continuous and either even or odd under the antipodal map, then there is a rotation $\rho \in \mathrm{SO}(3)$ satisfying* (1).

The proof is a configuration-space/test-map argument. Let

$$X_N = \mathrm{Fr}_O(TN)/D_4,$$

where $\mathrm{Fr}_O(TN)$ is the full orthonormal frame bundle and $D_4$ is the dihedral group of order eight. The space $X_N$ is a closed three-manifold fibered over $N$ with circle fiber. The four transported values define a section of a rank-three real vector bundle $\mathcal{E}_\mathcal{L} \to X_N$, and its zeros correspond to $\mathcal{L}$-tables modulo the symmetries of the square. We prove that

$$\langle w_3(\mathcal{E}_\mathcal{L}), [X_N]_2 \rangle = \chi(N) \pmod 2,$$

where $[X_N]_2$ is the mod 2 fundamental class. Thus the obstruction is independent of $\mathcal{L}$ and is nonzero when $\chi(N)$ is odd.

The use of a nonzero third Stiefel–Whitney class relates this argument to those of Hausel, Makai, and Szűcs [4] and Vassiliev [9]. Those proofs evaluate the obstruction geometrically using zeros of a suitable section. Here the evaluation follows from a bundle splitting, the Wu formula for surfaces, and integration along the circle fiber.

This note is organized as follows. Section 2 defines tables with values in a line bundle and derives the two corollaries from Theorem 1.1. Section 3 constructs and splits the test bundle. Section 4 computes its top Stiefel–Whitney class and proves the theorem.

## 2 Tables with values in a line bundle

### 2.1 Radial transport and twisted tables

Let $(N, g)$ be a closed connected Riemannian surface, and let $\mathcal{L} \to N$ be a smooth real line bundle equipped with a smooth connection. We use the associated parallel transport; the connection need not preserve a fiber metric.

Fix $a > 0$. For an orthonormal frame

$$p = (x; e_1, e_2) \in \mathrm{Fr}_O(TN),$$

put

$$v_0 = e_1, \qquad v_1 = e_2, \qquad v_2 = -e_1, \qquad v_3 = -e_2.$$

Let

$$x_i = \exp_x(a v_i), \qquad \gamma_i(t) = \exp_x(t a v_i), \quad 0 \le t \le 1.$$

For a continuous section $s$ of $\mathcal{L}$, define its radially transported values by

$$c_i(p; s) = \mathrm{PT}_{\gamma_i}^{-1}\big(s(x_i)\big) \in \mathcal{L}_x. \tag{2}$$

All four values now lie in the same one-dimensional vector space and depend continuously on the frame $p$.

**Definition 2.1.** The frame $p = (x; e_1, e_2)$ is an *$\mathcal{L}$-table of radius $a$* for $s$ if

$$c_0(p; s) = c_1(p; s) = c_2(p; s) = c_3(p; s).$$

The radius is measured in $T_xN$. We impose no restriction involving the injectivity radius: the radial geodesics are part of the construction, and the four endpoints need not be distinct.

*Remark* 2.2. For the trivial line bundle with the trivial connection, (2) is simply

$$c_i(p; s) = s(x_i) \in \mathbb{R}.$$

Thus Definition 2.1 reduces to Naseri Sadr's definition of a table [7]. We call an $\mathcal{L}$-table in this situation an *exponential table*.

## 2.2 The double-cover model

Let

$$q : \widetilde{N} \longrightarrow N$$

be a Riemannian double cover with deck involution $\tau$. Its associated sign line bundle is

$$\mathcal{L}_q = \widetilde{N} \times_{\mathbb{Z}/2} \mathbb{R}_{\mathrm{sgn}} \longrightarrow N.$$

We equip it with its canonical flat connection. A continuous odd function

$$h : \widetilde{N} \longrightarrow \mathbb{R}, \qquad h \circ \tau = -h,$$

defines a section $s_h$ of $\mathcal{L}_q$ by

$$s_h(q(\widetilde{x})) = [\widetilde{x}, h(\widetilde{x})].$$

This construction is a bijection between continuous anti-invariant functions on $\widetilde{N}$ and continuous sections of $\mathcal{L}_q$.

**Proposition 2.3.** *Choose a lift $\widetilde{x}$ of $x$ and lift the frame $(e_1, e_2)$ to $T_{\widetilde{x}}\widetilde{N}$. Let $\widetilde{x}_i$ be the endpoints of the lifts of the radial paths $\gamma_i$ beginning at $\widetilde{x}$. Then*

$$p \text{ is an } \mathcal{L}_q\text{-table for } s_h \quad \Longleftrightarrow \quad h(\widetilde{x}_0) = h(\widetilde{x}_1) = h(\widetilde{x}_2) = h(\widetilde{x}_3).$$

*Proof.* The choice of $\widetilde{x}$ identifies the fiber $(\mathcal{L}_q)_x$ with $\mathbb{R}$. Under this identification, flat transport along $\gamma_i$ is described by lifting $\gamma_i$ from $\widetilde{x}$. Consequently the element $c_i(p; s_h)$ corresponds exactly to the real number $h(\widetilde{x}_i)$. Replacing $\widetilde{x}$ by $\tau\widetilde{x}$ changes all four numbers by the same factor $-1$, so the equality condition is independent of the chosen lift. □

*Proof of Corollary 1.2 assuming Theorem 1.1.* The quotient map

$$q : \widetilde{N} \longrightarrow N = \widetilde{N}/\tau$$

is a Riemannian double cover. If $h \circ \tau = h$, then $h$ descends to an ordinary real-valued function on $N$. Apply Theorem 1.1 to the trivial line bundle with the trivial connection, and lift the resulting radial geodesics to $\widetilde{N}$.

If $h \circ \tau = -h$, then $h$ corresponds to a section of the sign line bundle $\mathcal{L}_q$. Apply Theorem 1.1 to $\mathcal{L}_q$. Proposition 2.3 identifies the resulting twisted table with an exponential table for $h$ on $\widetilde{N}$. □

*Proof of Corollary 1.4 assuming Theorem 1.1.* For the antipodal cover $q : S^2 \longrightarrow \mathbb{RP}^2$ we have $\chi(\mathbb{RP}^2) = 1$. Corollary 1.2 therefore gives an exponential table for every radius, in both parity cases.

For the standard metric of $S^2 \subset \mathbb{R}^3$,

$$\exp_x(ae) = \cos(a)x + \sin(a)e.$$

Thus the four exponential vertices are

$$\cos(a)x \pm \sin(a)e_1, \qquad \cos(a)x \pm \sin(a)e_2.$$

For $0 < a \leq \pi/2$ they form a nondegenerate Euclidean square inscribed in $S^2$. Conversely, the plane of any such square has distance $c \in [0, 1)$ from the origin. Choosing $a = \arccos(c)$ shows that its vertex set is a rotated copy of one of these configurations. Equality of the four function values is independent of their cyclic labeling, so this proves the corollary. □

# 3 The configuration space and the test bundle

## 3.1 The dihedral quotient

Let

$$\pi_P \colon P = \mathrm{Fr}_O(TN) \longrightarrow N$$

be the full orthonormal frame bundle. Let $D_4$ denote the dihedral group of order 8, realized as the following subgroup of $O(2)$:

$$D_4 = \langle R, S \mid R^4 = S^2 = 1,\ SRS = R^{-1} \rangle,$$

where

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The right action of $D_4$ on $P$ is free. Define

$$X_N = P/D_4, \qquad \pi : X_N \longrightarrow N.$$

The fiber is

$$O(2)/D_4 \cong S^1,$$

so $X_N$ is a closed three-manifold.

Let

$$\mathcal{W} = \Big\{ (t_0, t_1, t_2, t_3) \in \mathbb{R}^4 \mid t_0 + t_1 + t_2 + t_3 = 0 \Big\}$$

be the reduced permutation representation of $D_4$, defined by

$$R(t_0, t_1, t_2, t_3) = (t_3, t_0, t_1, t_2),$$
$$S(t_0, t_1, t_2, t_3) = (t_0, t_3, t_2, t_1).$$

The four directions $v_0, v_1, v_2, v_3$ are permuted by $R$ and $S$, and therefore so are the four transported values in (2).

Let $\mathcal{L}$ be a line bundle on $N$ equipped with a connection, and fix a continuous section $s \colon N \to \mathcal{L}$. For $p \in P$ and $x = \pi_P(p)$, put

$$\overline{c}(p; s) = \frac{1}{4} \sum_{i=0}^{3} c_i(p; s) \in \mathcal{L}_x,$$
$$\widetilde{\Phi}_s(p) = (c_0 - \overline{c}, \ldots, c_3 - \overline{c}) \in \mathcal{W} \otimes \mathcal{L}_x.$$

We use the associated-bundle convention $(pg, w) \sim (p, gw)$. The right action on frames gives $\widetilde{\Phi}_s(pg) = g^{-1}\widetilde{\Phi}_s(p)$, so this construction defines a section

$$\Phi_s \in \Gamma(X_N; \mathcal{E}_{\mathcal{L}})$$

of the rank-three bundle

$$\mathcal{E}_{\mathcal{L}} = (P \times_{D_4} \mathcal{W}) \otimes \pi^*\mathcal{L} \longrightarrow X_N. \tag{3}$$

**Proposition 3.1.** *The zero set of $\Phi_s$ parametrizes the $D_4$-orbits of $\mathcal{L}$-tables for $s$:*

$$\Phi_s([p]) = 0 \quad \Longleftrightarrow \quad c_0(p;s) = c_1(p;s) = c_2(p;s) = c_3(p;s).$$

*Proof.* The linear projection $\mathcal{L}_x^4 \to \mathcal{W} \otimes \mathcal{L}_x$ obtained by subtracting the mean of the four entries has the diagonal copy of $\mathcal{L}_x$ as its kernel. □

### 3.2 Splitting the untwisted bundle

Write

$$\mathcal{E}_0 = P \times_{D_4} \mathcal{W}.$$

There is a $D_4$-invariant decomposition

$$\mathcal{W} = U \oplus \ell,$$

where

$$U = \{(a_0, a_1, -a_0, -a_1) : a_0, a_1 \in \mathbb{R}\}, \qquad \ell = \mathbb{R}(1, -1, 1, -1).$$

Identifying $U$ with $\mathbb{R}^2$ via the coordinates $(a_0, a_1)$, the two generators act by

$$R|_U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S|_U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the coordinates $(a_0, a_1)$ identify $U$ with the standard representation of $D_4 \subset O(2)$. Writing $p = (x; e_1, e_2) \in P$, we obtain a vector bundle isomorphism

$$P \times_{D_4} U \longrightarrow \pi^*TN,$$
$$[p, (a_0, a_1, -a_0, -a_1)] \longmapsto ([p], a_0 e_1 + a_1 e_2).$$

This map is well defined because the identification $U \cong \mathbb{R}^2$ is $D_4$-equivariant and the right action on frames is given by composition.

On $\ell$, the action is the character

$$\lambda : D_4 \longrightarrow \{\pm 1\}, \qquad \lambda(R) = -1, \qquad \lambda(S) = +1.$$

Let

$$L_\lambda = P \times_{D_4} \mathbb{R}_\lambda.$$

We have proved the following splitting.

**Proposition 3.2.** *There are natural bundle isomorphisms*

$$\mathcal{E}_0 \cong \pi^*TN \oplus L_\lambda, \tag{4}$$
$$\mathcal{E}_{\mathcal{L}} \cong \pi^*(TN \otimes \mathcal{L}) \oplus (L_\lambda \otimes \pi^*\mathcal{L}). \tag{5}$$

*Proof.* The first isomorphism follows from the decomposition $\mathcal{W} = U \oplus \ell$. Tensoring (4) with $\pi^*\mathcal{L}$ gives the second. □

### 3.3 The fiber class

Put

$$\alpha = w_1(L_\lambda) \in \mathrm{H}^1(X_N; \mathbb{F}_2).$$

Here and below,

$$\pi_! : H^k(X_N; \mathbb{F}_2) \longrightarrow H^{k-1}(N; \mathbb{F}_2)$$

denotes integration along the circle fiber with mod 2 coefficients. No orientability assumption is needed with these coefficients.

**Lemma 3.3.** *The restriction of $L_\lambda$ to each circle fiber of $\pi : X_N \to N$ is the Möbius line bundle. Equivalently,*

$$\pi_!(\alpha) = 1 \in \mathrm{H}^0(N; \mathbb{F}_2).$$

*Proof.* Fix a frame over $x$ and rotate it through angles $0 \le \theta \le \pi/2$. In $X_N$ this is a loop, because the final frame is obtained from the initial one by $R$. The fibers of $L_\lambda$ at the two endpoints are glued by

$$\lambda(R) = -1.$$

Thus the restriction is Möbius, so $\alpha$ evaluates to one on the fiber. □

## 4 The characteristic-class calculation

We now show that the top Stiefel–Whitney class of $\mathcal{E}_\mathcal{L}$ is independent of $\mathcal{L}$. We use the standard properties of Stiefel–Whitney classes and the Wu formula; see [6, §§4 and 11].

All cohomology in this section has coefficients in $\mathbb{F}_2$, and juxtaposition denotes the cup product. For a closed $m$-manifold $M$, we write $[M]_2 \in H_m(M; \mathbb{F}_2)$ for its mod 2 fundamental class.

**Lemma 4.1.** *Let $E \to N$ be a real rank-two vector bundle and let $L \to N$ be a real line bundle with $u = w_1(L)$. Then*

$$w_1(E \otimes L) = w_1(E),$$

*and*

$$w_2(E \otimes L) = w_2(E) + u\, w_1(E) + u^2.$$

*Proof.* By the splitting principle, write the formal Stiefel–Whitney roots of $E$ as $t_1, t_2$. The roots of $E \otimes L$ are $t_1 + u, t_2 + u$. Therefore

$$(t_1 + u) + (t_2 + u) = t_1 + t_2$$

and

$$(t_1 + u)(t_2 + u) = t_1 t_2 + u(t_1 + t_2) + u^2.$$

□

**Lemma 4.2.** *For every $u \in \mathrm{H}^1(N; \mathbb{F}_2)$,*

$$u^2 = u\, w_1(TN) \in \mathrm{H}^2(N; \mathbb{F}_2).$$

*Proof.* Since $\mathrm{Sq}^1(u) = u^2$ and the first Wu class of $N$ is $v_1(N) = w_1(TN)$, the Wu formula [6, Theorem 11.14] gives

$$\left\langle u^2, [N]_2 \right\rangle = \langle u\, w_1(TN), [N]_2 \rangle .$$

The group $\mathrm{H}^2(N; \mathbb{F}_2)$ is one-dimensional because $N$ is closed and connected, so equality of the evaluations implies equality of the classes. □

**Proposition 4.3.** *For every real line bundle $\mathcal{L}$ on $N$,*

$$w_3(\mathcal{E}_{\mathcal{L}}) = \pi^* w_2(TN) \smile \alpha,$$

*where $\alpha = w_1(L_\lambda)$. Consequently,*

$$\langle w_3(\mathcal{E}_{\mathcal{L}}), [X_N]_2 \rangle = \chi(N) \pmod 2.$$

*Proof.* Put $u = w_1(\mathcal{L})$. By (5) and Lemma 4.1,

$$w_2\big(\pi^*(TN \otimes \mathcal{L})\big) = \pi^*\big(w_2(TN) + u\, w_1(TN) + u^2\big).$$

Moreover,

$$w_1\big(L_\lambda \otimes \pi^*\mathcal{L}\big) = \alpha + \pi^* u.$$

The top Stiefel–Whitney class of a rank-two bundle plus a line bundle is the product of the second class of the first summand with the first class of the second. Hence

$$\begin{aligned} w_3(\mathcal{E}_{\mathcal{L}}) &= \pi^*\big(w_2(TN) + u\, w_1(TN) + u^2\big) \smile (\alpha + \pi^* u) \\ &= \pi^*\big(w_2(TN) + u\, w_1(TN) + u^2\big) \smile \alpha. \end{aligned}$$

The term containing $\pi^* u$ vanishes because it is pulled back from a degree-three class on the surface $N$. Lemma 4.2 cancels the two $u$-dependent terms and gives

$$w_3(\mathcal{E}_{\mathcal{L}}) = \pi^* w_2(TN) \smile \alpha.$$

Using Lemma 3.3 and integration along the fiber,

$$\begin{aligned} \langle \pi^* w_2(TN) \smile \alpha, [X_N]_2 \rangle &= \langle w_2(TN) \smile \pi_!(\alpha), [N]_2 \rangle \\ &= \langle w_2(TN), [N]_2 \rangle \\ &= \chi(N) \pmod 2. \end{aligned}$$

The last equality is the mod 2 Euler characteristic formula [6, Corollary 11.12]. □

*Proof of Theorem 1.1.* Let $\Phi_s$ be the test section of $\mathcal{E}_{\mathcal{L}} \to X_N$. If $s$ had no $\mathcal{L}$-table of radius $a$, then Proposition 3.1 would imply that $\Phi_s$ is nowhere vanishing. A nowhere-zero section of a rank-three vector bundle splits off a trivial line and therefore forces its third Stiefel–Whitney class to vanish. On the other hand, Proposition 4.3 gives

$$\langle w_3(\mathcal{E}_{\mathcal{L}}), [X_N]_2 \rangle \equiv \chi(N) \equiv 1 \pmod 2.$$

This contradiction proves the theorem. □

## References


[1] R. İnanç Baykur, R. C. Kirby, and D. Ruberman (eds.), *K3: A New Problem List in Low-Dimensional Topology*, Mathematical Surveys and Monographs, vol. 295, American Mathematical Society, Providence, RI, 2026.

[2] M. C. Crabb and J. Jaworowski, *Theorems of Kakutani and Dyson revisited*, J. Fixed Point Theory Appl. **5** (2009), no. 2, 227–236. doi:10.1007/s11784-008-0099-9.

[3] F. J. Dyson, *Continuous functions defined on spheres*, Ann. of Math. (2) **54** (1951), no. 3, 534–536. doi:10.2307/1969487.

[4] T. Hausel, E. Makai, Jr., and A. Szűcs, *Inscribing cubes and covering by rhombic dodecahedra via equivariant topology*, Mathematika **47** (2000), no. 1–2, 371–397. doi:10.1112/S0025579300015965. arXiv:math/9906066.

[5] B. Matschke, *A survey on the square peg problem*, Notices Amer. Math. Soc. **61** (2014), no. 4, 346–352. doi:10.1090/noti1100.

[6] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, no. 76, Princeton University Press, Princeton, NJ, 1974.

[7] A. Naseri Sadr, *A table theorem for surfaces with odd Euler characteristic*, preprint, 2024, revised 2026. arXiv:2412.01977v3.

[8] P. Soberón, *A counterexample to the table conjecture on the sphere*, Hexagon preprint, 2026. hexagon:2610.00484v1.

[9] V. A. Vassiliev, *Inscribed squares of level sets of functions on the sphere*, preprint, 2026. arXiv:2607.13052v2.