A joint fourth cumulant bound for the stationary solution of a fractional SDE and the LAN property for $0 < H \leq 1/4$

Haruka Kyokaidori*†

October 6, 2026

Abstract

Let $(\bar{X}_t)_{t\in\mathbb{R}}$ be the stationary solution for an additive stochastic differential equation driven by a fractional Brownian motion. We give an upper bound for a joint fourth cumulant of the form $\kappa_4(f_0(\bar{X}_0), f_1(\bar{X}_{t_1}), f_2(\bar{X}_{t_2}), f_3(\bar{X}_{t_3}))$. Our tool is the partial Malliavin calculus for the stationary solution for an SDE driven by fBM. As an application, we show that local asymptotic normality of a likelihood ratio field for the drift parameter of a fractional SDE holds even in the case $0 < H \leq 1/4$.

Keywords: Malliavin calculus; fractional Brownian motion; joint fourth cumulant; local asymptotic normality; stationary solution

Contents

1 Introduction **2**

2 Main results **3**

3 Partial Malliavin calculus and the stationary solution **6**

3.1 Partial Malliavin calculus 6

3.2 The stationary solution 13

4 Cumulant Calculation **18**

4.1 A representation for the joint fourth cumulant 18

4.2 Application to the stationary solution 23

4.3 Bounds for Malliavin inner products 27

5 The LAN Property **33**

*E-mail: churchstreet@atamath.com

†Author note. **Haruka Kyokaidori** is a research persona of *the ATAMATH Project*, an experiment in mathematical research and exposition using fictional characters. **Kohei Chiba**, an independent researcher, is the responsible human researcher for this work and assumes responsibility for its mathematical content, references, revisions, and correspondence. The above address can also be used to contact the responsible human researcher.

Contact Information. The ATAMATH Project (https://atamath.com/); Kohei Chiba (https://sites.google.com/view/kchiba/)

1 Introduction

Let $(B_t)_{t\in\mathbb{R}}$ be a two-sided fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$. We suppress H in the notation for notational simplicity. We consider a one-dimensional stochastic differential equation

$$X_t^{\theta,x} = x + \int_0^t a(X_s^{\theta,x}, \theta)ds + \sigma B_t, \quad t \geq 0, \quad x \in \mathbb{R}, \quad \theta \in \Theta, \quad \sigma \neq 0. \tag{1}$$

Here Θ is an open subset of $\mathbb{R}^m$. It is known that the equation (1) has a stationary solution $(\bar{X}_t^\theta)_{t\in\mathbb{R}}$ satisfying

$$\bar{X}_t^\theta = \bar{X}_s^\theta + \int_s^t a(\bar{X}_r^\theta, \theta)dr + \sigma(B_t - B_s), \quad s < t, \quad \theta \in \Theta, \quad \sigma \neq 0 \tag{2}$$

under Assumption 1. Our stationary solution is based on a random dynamical system [Garrido-Atienza et al., 2009] (see also Chiba [2018, 2020]).

Stochastic differential equations driven by additive fractional Brownian noise have been investigated from several perspectives in probability and statistics, under various assumptions on the drift and the Hurst parameter. On the statistical side, examples include drift parameter estimation [Kleptsyna and Le Breton, 2002, Neuenkirch and Tindel, 2014, Chiba, 2020] and estimation of the stationary density [Amorino et al., 2026]. On the probabilistic side, these systems have been studied in connection with ergodicity [Hairer, 2005], numerical approximation of stationary solutions [Garrido-Atienza et al., 2009], density regularity [Li et al., 2023], and averaging principles [Li and Sieber, 2022].

Among them, in view of likelihood analysis, Liu et al. [2019] and Chiba [2018] studied the local asymptotic normality (LAN) of a likelihood ratio field for a drift parameter θ under continuous long-time observation asymptotics when $1/2 < H < 1$ and $1/4 < H < 1/2$, respectively.

In the proof of the LAN property when $0 < H < 1/2$, we need to bound a product-covariance of the form

$$\mathrm{Cov}\{(\partial_{\theta_i} a)(\bar{X}_{r_1}^\theta)(\partial_{\theta_j} a)(\bar{X}_{u_1}^\theta), (\partial_{\theta_i} a)(\bar{X}_{r_2}^\theta)(\partial_{\theta_j} a)(\bar{X}_{u_2}^\theta)\}$$

with $\mathbb{E}\{(\partial_{\theta_i} a)(\bar{X}_0^\theta)\} = \mathbb{E}\{(\partial_{\theta_j} a)(\bar{X}_0^\theta)\} = 0$, which naturally reduces to a (probabilistic) problem of bounding a joint fourth cumulant, say

$$\kappa_4(f(\bar{X}_0^\theta), g(\bar{X}_{u_1-r_1}^\theta), f(\bar{X}_{r_2-r_1}^\theta), g(\bar{X}_{u_2-r_1}^\theta))$$

with $\mathbb{E}\{f(\bar{X}_0^\theta)\} = \mathbb{E}\{g(\bar{X}_0^\theta)\} = 0$. Chiba [2018, 2020] bound the product-covariance using Poincaré's inequality and this yields the restriction $1/4 < H < 1/2$. On the other hand, Chiba and Takabatake [2024] treat the same problem in the fOU case $a(x, \theta) = -\theta x \quad (\theta > 0)$, and thanks to Gaussianity (Isserlis' theorem), show the LAN property for all $0 < H < 1/2$. This suggests that with a more precise bound on a joint fourth cumulant we may show the LAN property for a general drift coefficient $a(x, \theta)$ even when $0 < H \leq 1/4$. This is the original motivation for considering the joint fourth cumulant, and the suggested extension is indeed possible.

This paper is organized as follows. In section 2, we summarize theorems to be proved and assumptions which are assumed throughout the paper. Section 3 presents some auxiliary results on partial Malliavin calculus and its applications to the stationary solution. We bound the joint fourth cumulant in Section 4. Finally, the bound for the cumulant is applied to the proof of the LAN property.

Remark. In view of statistical applications, we formulate the results with a parametrized model (1). If we consider purely probabilistic situations, it suffices to just suppress θ in the notation.

Notation. In this paper, an integral $\int f(r)dr$ is sometimes written in the operator form $\int dr\ f(r)$.

2 Main results

We impose the following assumptions on the drift coefficient $a : \mathbb{R} \times \Theta \to \mathbb{R}$.

Assumption 1 (Assumptions for a cumulant bound)**.**

(A1) Partial derivatives $\partial_x a(x,\theta), \partial_x^2 a(x,\theta), \partial_x^3 a(x,\theta)$ exist, are continuous and bounded in x for each $\theta \in \Theta$. Furthermore, $x \mapsto \partial_x a(x,\theta)$ is negative: there exist positive numbers $\alpha = \alpha(\theta)$ and $\alpha' = \alpha'(\theta)$ such that

$$-\alpha' \le \partial_x a(\cdot,\theta) \le -\alpha$$

for each θ.

In the case of $a(x,\theta) = a(x)$, we just drop θ's in the assumption $(A1)$.

Assumption 2 (Assumptions for the LAN Property)**.** There exists a neighborhood U of θ such that the following property holds.

(A2) A partial derivative $(\partial_\theta a)(x,\theta)$ exists and has polynomial growth: there exists a positive constant $C = C(U)$ and a non-negative integer $p = p(U)$ such that

$$\sup_{\theta' \in U} |(\partial_\theta a)(x,\theta')| \le C(1 + |x|^p)$$

and

$$|(\partial_\theta a)(x,\theta_1) - (\partial_\theta a)(x,\theta_2)| \le C(1 + |x|^p)|\theta_1 - \theta_2|$$

for all $x \in \mathbb{R}$ and $\theta_1, \theta_2 \in U$.

(A3) For each $\theta' \in U$, the partial derivatives $\partial_x \partial_\theta a(\cdot,\theta')$, $\partial_x^2 \partial_\theta a(\cdot,\theta')$ and $\partial_x^3 \partial_\theta a(\cdot,\theta')$ exist, are continuous and bounded in x.

Notation. Let $H \in (0, 1/2)$ and let $C_c^\infty(\mathbb{R})$ denote the space of smooth functions with compact support.

(1) Fractional integral $I_\pm^{1/2-H}$. For $\varphi \in C_c^\infty(\mathbb{R})$, we define a fractional integral of order $1/2 - H$ by

$$(I_\pm^{1/2-H}\varphi)(x) = \frac{1}{\Gamma(1/2-H)} \int_0^\infty t^{-H-1/2}\varphi(x \mp t)\ dt \quad (x \in \mathbb{R}).$$

The operator $I_\pm^{1/2-H}$ can be extended to a bounded operator from $L^2(\mathbb{R})$ to $L^{1/H}(\mathbb{R})$ and the extended operator is also denoted by $I_\pm^{1/2-H}$. See Samko et al. [1993, Theorem 5.3].

(2) Fractional derivative $\mathbf{D}_\pm^{1/2-H}$. Let $\psi \in I_\pm^{1/2-H}(L^2(\mathbb{R}))$. Its (Marchaud) fractional derivative of order $1/2 - H$ is

$$(\mathbf{D}_\pm^{1/2-H}\psi)(x) = L^2\text{-}\lim_{\epsilon \to 0} \frac{1/2-H}{\Gamma(H+1/2)} \int_\epsilon^\infty \frac{\psi(x) - \psi(x \mp t)}{t^{3/2-H}}\, dt \quad (x \in \mathbb{R}).$$

By Samko et al. [1993, Theorem 6.1], we have $\mathbf{D}_-^{1/2-H} I_-^{1/2-H} f = f$ for $f \in L^2(\mathbb{R})$.

(3) Hilbert space $\mathcal{H}$. Let $\mathcal{H}$ denote the space $I_-^{1/2-H}(L^2(\mathbb{R}))$ with inner product

$$\langle f, g \rangle_{\mathcal{H}} = e_H \langle \mathbf{D}_-^{1/2-H} f, \mathbf{D}_-^{1/2-H} g \rangle_{L^2(\mathbb{R})}$$

with $e_H = \Gamma(2H+1)\sin(\pi H)$. A fractional Brownian motion B defines an isonormal Gaussian process over $\mathcal{H}$ (see Pipiras and Taqqu, 2000).

(4) Partial Malliavin derivative. Let $W^{(i)}$ be isonormal Gaussian processes over $\mathcal{H}$ on a complete probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ for $i = 1, \dots, n$. We can assume $\mathcal{F}_i$ is the completion of the σ-algebra generated by $W^{(i)}$. For $(c_1, \dots, c_n) \in \mathbb{R}^n$ with $\sum_i |c_i|^2 = 1$, we set

$$W^c(h)(\omega_1, \dots, \omega_n) := \sum_{i=1}^n c_i W^{(i)}(h)(\omega_i) \quad (\omega_i \in \Omega_i).$$

Then W^c defines an isonormal Gaussian process on $(\prod_i \Omega_i, \otimes_i \mathcal{F}_i, \otimes_i \mathbb{P}_i)$. For a functional of W^c, we denote by $\mathbb{E}_i$ the expectation with respect to $\mathbb{P}_i$ and by $D^{(i)}$ the Malliavin derivative with respect to $W^{(i)}$.

Let us give the partial Malliavin derivatives of the stationary solution. They are useful tools to investigate the joint fourth cumulant.

Theorem 3. *Suppose that Assumption 1 is fulfilled. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting independent two-sided fBMs $B^{(1)}, \dots, B^{(n)}$. For $c \in \mathbb{R}^n$, $|c| = 1$, the process $B^c = \sum_{i=1}^n c_i B^{(i)}$ is also a two-sided fBM on $(\Omega, \mathcal{F}, \mathbb{P})$.*

(1) *There is a unique stationary process $Y^{c,\theta} = (Y_t^{c,\theta})_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying*

$$Y_t^{c,\theta} - Y_s^{c,\theta} = \int_s^t a(Y_r^{c,\theta}, \theta)\, dr + \sigma(B_t^c - B_s^c) \tag{3}$$

for all $s < t$, $\mathbb{P}$-almost surely.

We set $J_{t,s}^{c,\theta} = \exp\left(\int_s^t (\partial_x a)(Y_r^{c,\theta}, \theta) dr\right) \mathbf{1}_{(-\infty,t]}(s)$ for $s, t \in \mathbb{R}$.

(2) *For any $t \in \mathbb{R}$, the random variable $Y_t^{c,\theta}$ has partial Malliavin derivatives up to order three, and its Malliavin derivatives are given by*

$$\begin{aligned} D_{s_1}^{(i_1)} Y_t^{c,\theta} &= \sigma c_{i_1} J_{t,s_1}^{c,\theta}, \\ D_{s_2}^{(i_2)} D_{s_1}^{(i_1)} Y_t^{c,\theta} &= \int_{-\infty}^t (\partial_x^2 a)(Y_r^{c,\theta}, \theta) J_{t,r}^{c,\theta} (D_{s_1}^{(i_1)} Y_r^{c,\theta})(D_{s_2}^{(i_2)} Y_r^{c,\theta})\, dr, \end{aligned}$$

and

$$\begin{aligned} D_{s_3}^{(i_3)} D_{s_2}^{(i_2)} D_{s_1}^{(i_1)} Y_t^{c,\theta} = \int_{-\infty}^t J_{t,r}^{c,\theta} \Big[& \partial_x^3 a(Y_r^{c,\theta}, \theta)(D_{s_1}^{(i_1)} Y_r^{c,\theta})(D_{s_2}^{(i_2)} Y_r^{c,\theta})(D_{s_3}^{(i_3)} Y_r^{c,\theta}) \\ & + \partial_x^2 a(Y_r^{c,\theta}, \theta) \Big\{ (D_{s_3}^{(i_3)} D_{s_1}^{(i_1)} Y_r^{c,\theta})(D_{s_2}^{(i_2)} Y_r^{c,\theta}) + (D_{s_3}^{(i_3)} D_{s_2}^{(i_2)} Y_r^{c,\theta})(D_{s_1}^{(i_1)} Y_r^{c,\theta}) \\ & + (D_{s_1}^{(i_1)} D_{s_2}^{(i_2)} Y_r^{c,\theta})(D_{s_3}^{(i_3)} Y_r^{c,\theta}) \Big\} \Big] dr. \end{aligned}$$

Furthermore, these partial Malliavin derivatives satisfy

$$\begin{aligned} & \mathbb{E}\{|Y_t^{c,\theta}|^p\} + \mathbb{E}\{\|D^{(i_1)} Y_t^{c,\theta}\|_{\mathcal{H}}^p\} \\ & + \mathbb{E}\{\|D^{(i_2)} D^{(i_1)} Y_t^{c,\theta}\|_{\mathcal{H}^{\otimes 2}}^p\} + \mathbb{E}\{\|D^{(i_3)} D^{(i_2)} D^{(i_1)} Y_t^{c,\theta}\|_{\mathcal{H}^{\otimes 3}}^p\} < \infty. \end{aligned}$$

for $p \geq 1$ and $i_1, i_2, i_3 \in \{1, \dots, n\}$.

Remark. Let $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ be the space of continuous functions on $\mathbb{R}$ with the σ-algebra associated with compact open topology. Note that the law of $Y^{c,\theta}$ and $\bar{X}^\theta$ coincide for any $|c| = 1$ on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$. In particular we use $Y^{c,\theta}$ instead of $\bar{X}^\theta$ for calculations involving expectation: $E\{F(\bar{X}^\theta)\} = E\{F(Y^{c,\theta})\}$ for a measurable function $F : C(\mathbb{R}) \to \mathbb{R}$ whenever the expectations are well defined.

We introduce a class of test functions for a joint fourth cumulant.

Notation. We set

$$\mathcal{F}_\theta = \left\{ f \in C^3(\mathbb{R}) \;\middle|\; \mathbb{E}\{f(\bar{X}_0^\theta)\} = 0, \sum_{i=0}^{3} |f^{(i)}(x)| \le C(1+|x|^p) \text{ for some } C, p > 0 \right\}.$$

Let $t_0 < t_1 < t_2 < t_3$ and $f_i : \mathbb{R} \to \mathbb{R}$ $(i = 0, \dots, 3)$ be measurable functions satisfying $\mathbb{E}\{f_i(\bar{X}_0^\theta)\} = 0$ and $\mathbb{E}|f_i(\bar{X}_0^\theta)|^4 < \infty$. Recall that the joint fourth cumulant $\kappa_4(f_0(\bar{X}_{t_0}^\theta), f_1(\bar{X}_{t_1}^\theta), f_2(\bar{X}_{t_2}^\theta), f_3(\bar{X}_{t_3}^\theta))$ is

$$\begin{aligned}
&\kappa_4(f_0(\bar{X}_{t_0}^\theta), f_1(\bar{X}_{t_1}^\theta), f_2(\bar{X}_{t_2}^\theta), f_3(\bar{X}_{t_3}^\theta)) \\
&= E\{f_0(\bar{X}_{t_0}^\theta) f_1(\bar{X}_{t_1}^\theta) f_2(\bar{X}_{t_2}^\theta) f_3(\bar{X}_{t_3}^\theta)\} - E\{f_0(\bar{X}_{t_0}^\theta) f_1(\bar{X}_{t_1}^\theta)\} E\{f_2(\bar{X}_{t_2}^\theta) f_3(\bar{X}_{t_3}^\theta)\} \\
&\quad - E\{f_0(\bar{X}_{t_0}^\theta) f_2(\bar{X}_{t_2}^\theta)\} E\{f_1(\bar{X}_{t_1}^\theta) f_3(\bar{X}_{t_3}^\theta)\} - E\{f_0(\bar{X}_{t_0}^\theta) f_3(\bar{X}_{t_3}^\theta)\} E\{f_1(\bar{X}_{t_1}^\theta) f_2(\bar{X}_{t_2}^\theta)\}.
\end{aligned}$$

Theorem 4. *Assume that Assumption 1 is fulfilled. Let $t_0, t_1, t_2, t_3 \in \mathbb{R}$ be distinct and $t_{(0)} < t_{(1)} < t_{(2)} < t_{(3)}$ be the increasing rearrangement of $t_0, \dots, t_3$. For functions $f_0, f_1, f_2, f_3 \in \mathcal{F}_\theta$, there exists a positive constant*

$$C = C(f_0, f_1, f_2, f_3, a, H, \sigma, \theta)$$

such that

$$|\kappa_4(f_0(\bar{X}_{t_0}^\theta), f_1(\bar{X}_{t_1}^\theta), f_2(\bar{X}_{t_2}^\theta), f_3(\bar{X}_{t_3}^\theta))| \le C \prod_{i=0}^{2} (1 + |t_{(i+1)} - t_{(i)}|)^{H-3/2}.$$

Now we proceed to the LAN property for the model (1). We recall a definition of the LAN property [Chiba, 2018, Definition 2.6]. We allow a positive semidefinite information matrix; nondegeneracy is not asserted.

Notation. Let m be a positive integer. For a vector $u \in \mathbb{R}^m$, we denote the transpose of u by $u^\star$. Let $\mu \in \mathbb{R}^m$ and $\Sigma \in \mathbb{R}^{m \times m}$ be a positive semi-definite matrix. The m-dimensional normal distribution with mean μ and variance Σ is denoted by $\mathcal{N}(\mu, \Sigma)$.

Definition 5 (LAN property)**.** Let $(\mu_\theta^T)_{\theta \in \Theta}$ be a family of probability measures that are mutually absolutely continuous. For a nondegenerate $m \times m$ matrix $\varphi_T(\theta)$, we set

$$Z_\theta^T(u) = \frac{d\mu_{\theta + \varphi_T(\theta)u}^T}{d\mu_\theta^T}.$$

A family $(\mu_\theta^T)_{\theta \in \Theta}$ is called locally asymptotically normal (LAN) at a point $\theta \in \Theta$ if $Z_\theta^T(u)$ can be represented as

$$Z_\theta^T(u) = \exp\left(u^\star \Delta_\theta^T - \frac{1}{2} u^\star I(\theta) u + r_{\theta,u}^T \right).$$

Here $I(\theta)$ is a positive semi-definite $m \times m$ matrix, the random variable Δ_θ^T converges in distribution (with respect to μ_θ^T) to $\mathcal{N}_m(0, I(\theta))$ as $T \to \infty$, and $r_{\theta,u}^T$ satisfies

$$\lim_{T \to \infty} \mu_\theta^T \{ |r_{\theta,u}^T| > \epsilon \} = 0$$

for any $\epsilon > 0$ and each fixed $u \in \mathbb{R}^m$, with $\theta + \varphi_T(\theta)u \in \Theta$ for all sufficiently large T.

Notation. The $m \times m$ identity matrix is denoted by E_m and the Beta function by $B(x, y)$. We set a constant $\bar{d}_H$ which appears in the Fisher information:

$$\bar{d}_H = \sqrt{2HB(3/2 - H, 1/2 - H)B(1/2 + H, 1/2 - H)}.$$

For the model (1), the LAN property is formulated as follows. Let $C([0, T])$ be the space of continuous functions on $[0, T]$. We consider the sup-norm on $C[0, T]$ and the associated Borel σ-algebra is denoted by $\mathcal{B}(C([0, T]))$. Let μ_θ^T be the law of the solution $(X_t^{\theta,x})_{t\in[0,T]}$ on $C([0, T])$.

Theorem 6 (LAN property)**.** *Suppose that Assumptions 1 and 2 are fulfilled. Furthermore we assume* $\mathbb{E}\{(\partial_\theta a)(\bar{X}_0^\theta, \theta)\} = 0$. *Then the family* $(\mu_\theta^T)_{\theta\in\Theta}$ *is LAN at a point* $\theta \in \Theta$*, with a normalizing matrix*

$$\varphi_T(\theta) = T^{-1/2}E_m$$

and the Fisher information matrix

$$I(\theta) = \sigma^{-2}\bar{d}_H^{-2} \int_0^\infty \int_0^\infty r^{-H-1/2}u^{-H-1/2}\mathbb{E}\{(\partial_\theta a)(\bar{X}_r^\theta, \theta)(\partial_\theta a)(\bar{X}_u^\theta, \theta)^\star\}\, du\, dr.$$

Remark. For a scalar parameter with $\mathbb{E}\{(\partial_\theta a)(\bar{X}_0^\theta, \theta)\} \neq 0$, the score has the order T^{1-H} and the corresponding normalization is $T^{-(1-H)}$; see Chiba [2018, Theorem 2.8 and Section 5.1]. Here we restrict our attention to the centered case $\mathbb{E}\{(\partial_\theta a)(\bar{X}_0^\theta, \theta)\} = 0$, which has an essential difficulty and the condition $1/4 < H < 1/2$ was required in Chiba [2018].

3 Partial Malliavin calculus and the stationary solution

3.1 Partial Malliavin calculus

We introduce the partial Malliavin calculus in this section. The partial Malliavin calculus on product spaces has appeared in more general forms in Kusuoka and Stroock [1984] and Nualart and Zakai [1989]. Since only a simple tensorized version is needed here, we give a self-contained formulation adapted to our setting. In the following, all appearing σ-algebras are assumed to be completed.

Settings. Let $W^{(i)}$ be isonormal Gaussian processes associated with a Hilbert space $\mathcal{H}_i$ on a complete probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ for $i = 1, \ldots, n$. We can assume $\mathcal{F}_i$ is the σ-algebra generated by $W^{(i)}$. We set $(\Omega, \mathcal{F}, \mathbb{P}) = (\prod_{i=1}^n \Omega_i, \otimes_{i=1}^n \mathcal{F}_i, \otimes_{i=1}^n \mathbb{P}_i)$. An isonormal Gaussian process $(\omega_1, \ldots, \omega_n) \mapsto W^{(i)}(\omega_i)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is also denoted by $W^{(i)}$. We set an isonormal Gaussian process $W^\oplus$ over $\mathcal{K} = \oplus_{j=1}^n \mathcal{H}_j$ by

$$W^\oplus\left(\sum_{j=1}^n h_j\right) = \sum_{j=1}^n W^{(j)}(h_j).$$

Note that the σ-algebra $\sigma(W^\oplus)$ generated by $W^\oplus$ coincides with $\mathcal{F} = \sigma(W^{(1)}, \ldots, W^{(n)})$ generated by $W^{(1)}, \ldots, W^{(n)}$. Indeed, if we take $k = 0 \oplus \cdots \oplus h_i \oplus \cdots \oplus 0 \in \mathcal{K},\ h_i \in \mathcal{H}_i$, then $W^\oplus(k) = W^{(i)}(h_i)$.

Let $C_p^\infty(\mathbb{R}^m)$ denote the space of smooth functions satisfying $|\partial^\alpha f(x)| \leq C_\alpha(1 + |x|)^{N_\alpha}$ for every multi-index α. For a Hilbert space $\mathcal{H}$ and an isonormal Gaussian process W over $\mathcal{H}$, we define $\mathcal{S}_0(\Omega; V)$ to be a class of smooth random variables such that $F \in \mathcal{S}_0(\Omega; V)$ has the form

$$F = f(W(h_1), \ldots, W(h_m))v$$

where $f \in C_p^\infty(\mathbb{R}^m), v \in V$ and $h_i \in \mathcal{H}, i = 1\ldots, m$. We set $\mathcal{S}(\Omega; V) = \text{span } \mathcal{S}_0(\Omega; V)$. We define a projection $P_i : \mathcal{K} \to \mathcal{H}_i,\ \sum_{j=1}^n h_j \mapsto h_i$.

Notation. For a Hilbert space $\mathcal{H}$, the L^p-space of $\mathcal{H}$-valued functions is denoted by $L^p(\Omega; \mathcal{H})$. Note that $L^p(\Omega; \mathcal{H})$ consists of the strongly measurable $\mathcal{H}$-valued functions F satisfying $\mathbb{E}\{\|F\|_{\mathcal{H}}^p\} < \infty$. Similarly, we denote the V-valued Malliavin-Sobolev space by $\mathbb{D}^{r,p}(\Omega; V)$. All tensor products of Hilbert spaces are considered as completed. The algebraic tensor product of Hilbert spaces is denoted $\mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2 = \text{span}\{h_1 \otimes h_2 \mid h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2\}$.

Partial Malliavin derivative. Let V be a separable Hilbert space. We can consider an ordinary Malliavin derivative (with respect to $W^\oplus$) $D^r : \mathbb{D}^{r,p}(V) \to L^p(\Omega; V \otimes \mathcal{K}^r)$. Recall that for $F \in \mathcal{S}_0(\Omega; V)$

$$D^k F = \sum_{l_1,\ldots,l_k=1}^m \partial_{l_k} \cdots \partial_{l_1} f(W^\oplus(h_1), \ldots, W^\oplus(h_m)) v \otimes h_{l_1} \otimes \cdots \otimes h_{l_k}.$$

For a multiple index $\mathbf{k} = (\mathbf{k}_k, \ldots, \mathbf{k}_1),\ \mathbf{k}_i \in \{1, \ldots, n\}$, the partial Malliavin derivative $D^{\mathbf{k}} F$ is defined by

$$D^{\mathbf{k}} F = (I_V \otimes P_{\mathbf{k}_1} \otimes \cdots \otimes P_{\mathbf{k}_k})(D^k F).$$

Note that the $\mathbf{k}$-th Malliavin derivative of $F \in \mathcal{S}_0(\Omega; V)$ is defined by

$$D^{\mathbf{k}} F = \sum_{l_1,\ldots,l_k=1}^m \partial_{l_k} \cdots \partial_{l_1} f(W^\oplus(h_1), \ldots, W^\oplus(h_m)) v \otimes P_{\mathbf{k}_1} h_{l_1} \otimes \cdots \otimes P_{\mathbf{k}_k} h_{l_k}.$$

We set

$$\|F\|_{\mathbf{k},p;V} = \left(\mathbb{E}\{\|F\|_V^p\} + \mathbb{E}\{\|D^{(\mathbf{k}_1)} F\|_{V \otimes \mathcal{H}_{\mathbf{k}_1}}^p\} + \cdots + \mathbb{E}\{\|D^{(\mathbf{k}_k)} \cdots D^{(\mathbf{k}_1)} F\|_{V \otimes \mathcal{H}_{\mathbf{k}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{k}_k}}^p\}\right)^{1/p}.$$

for $1 \le p < \infty$ and $\mathbf{k} = (\mathbf{k}_k, \ldots, \mathbf{k}_1)$. The closure of $\mathcal{S}(\Omega; V)$ with respect to $\|\cdot\|_{\mathbf{k},p;V}$ is denoted by $\mathbb{D}^{\mathbf{k},p}$. Note that $\mathbb{D}^{k,p}$ denotes the ordinary Malliavin-Sobolev space.

We set $\omega_{-i} = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n)$ and $(\Omega_{-i}, \mathbb{P}_{-i}) = (\prod_{j \neq i} \Omega_j, \otimes_{j \neq i} \mathbb{P}_j)$. We can identify

$$L^2(\Omega; V) \simeq L^2(\Omega_{-i}) \otimes L^2(\Omega_i; V) \simeq L^2(\Omega_{-i} \times \Omega_i; V)$$

(see Reed and Simon, 1980, Theorem II.10). Let D_i be the Malliavin derivative on Ω_i with respect to $W^{(i)}$. An operator $I_{L^2(\Omega_{-i})} \widehat{\otimes} D_i$ denotes the closure of the operator defined on the algebraic tensor product by

$$(I_{L^2(\Omega_{-i})} \widehat{\otimes} D_i)(G_{-i} G_i) = G_{-i} D_i G_i, \qquad G_{-i} \in L^2(\Omega_{-i}), \qquad G_i \in \mathbb{D}^{1,2}(\Omega_i; V).$$

Proposition 7. *The closed partial Malliavin derivative $D^{(i)}$ (corresponding to $\mathbf{k} = (i)$) is identified with $I_{L^2(\Omega_{-i})} \widehat{\otimes} D_i$. Moreover, for $F \in L^2(\Omega; V)$,*

$$F \in \mathbb{D}^{(i),2}(\Omega; V) \quad \Leftrightarrow \quad F(\omega_{-i}, \cdot) \in \mathbb{D}^{1,2}(\Omega_i; V) \quad \text{for } \mathbb{P}_{-i}\text{-a.e. } \omega_{-i}, \text{ and} \\ \int_{\Omega_{-i}} \|D_i(F(\omega_{-i}, \cdot))\|_{L^2(\Omega_i; V \otimes \mathcal{H}_i)}^2 \, \mathbb{P}_{-i}(d\omega_{-i}) < \infty. \tag{4}$$

Proof. We set $\mathcal{K}_{-i} = \bigoplus_{j \neq i} \mathcal{H}_j$ and $\iota_{-i} : \mathcal{K}_{-i} \to \mathcal{K}$,

$$\iota_{-i}(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n) = (h_1, \ldots, h_{i-1}, 0, h_{i+1}, \ldots, h_n).$$

We define an isonormal Gaussian process $W^{\oplus,-i}$ over $\mathcal{K}_{-i}$ by $W^{\oplus,-i}(h) = W^{\oplus}(\iota_{-i}h)$. Then the random variables in $L^2(\Omega_{-i})$ are approximated by $\mathcal{S}(\Omega_{-i})$, and $G_{-i} \in \mathcal{S}(\Omega_{-i})$ is of the form

$$G_{-i} = f(W^{\oplus,-i}(h_1), \dots, W^{\oplus,-i}(h_m)) = f(W^{\oplus}(\iota_{-i}h_1), \dots, W^{\oplus}(\iota_{-i}h_m)), \qquad h_1, \dots, h_m \in \mathcal{K}_{-i}.$$

This gives $D^{(i)}G_{-i} = 0$.

For a simple random variable of the form $G = G_{-i}G_i$, we have

$$D^{(i)}G = (D^{(i)}G_{-i})G_i + G_{-i}(D^{(i)}G_i) = G_{-i}D_iG_i = (I_{L^2(\Omega_{-i})}\widehat{\otimes}D_i)G.$$

Since random variables of this type are dense in $L^2(\Omega_{-i} \times \Omega_i; V)$, we conclude $D^{(i)} = I_{L^2(\Omega_{-i})}\widehat{\otimes}D_i$. Note that the right hand side is defined on $L^2(\Omega_{-i}) \otimes \mathbb{D}^{1,2}(\Omega_i; V) = L^2(\Omega_{-i}; \mathbb{D}^{1,2}(\Omega_i; V))$, which gives the condition (4). □

The following statement follows immediately from the above proposition.

Corollary 8. *Let $i \in \{1, \dots, n\}$ and $F \in \mathbb{D}^{1,2}(\Omega; V)$. Then*

$$F(\omega_{-i}, \cdot) \in \mathbb{D}^{1,2}(\Omega_i; V), \qquad (D^{(i)}F)(\omega_{-i}, \cdot) = D_i(F(\omega_{-i}, \cdot)).$$

$\mathbb{P}_{-i}$-a.s.

Next we show $D^{(i)}$ and $\mathbb{E}_j$ commute if $i \neq j$. The expectation with respect to $\mathbb{P}_j$ is denoted $\mathbb{E}_j$:

$$\mathbb{E}_j\{F\} = \int_{\Omega_j} F(\omega_1, \dots, \omega_n)\mathbb{P}_j(d\omega_j).$$

Lemma 9. *Let $1 \leq p < \infty$, $m \leq k$ and $\mathbf{k} = (\mathbf{k}_m, \dots, \mathbf{k}_1)$ with $\mathbf{k}_l \neq j$ for all $l = 1, \dots, m$. If $F \in \mathbb{D}^{k,p}(\Omega; V)$, then $\mathbb{E}_j\{F\} \in \mathbb{D}^{k,p}(\Omega_{-j}; V)$ and $D^{\mathbf{k}}\mathbb{E}_j\{F\} = \mathbb{E}_j\{D^{\mathbf{k}}F\}$, $\mathbb{P}_{-j}$-a.s.*

Proof. For $F = f(W^{\oplus}(h_1), \dots, W^{\oplus}(h_d))v$, $h_1, \dots, h_d \in \mathcal{K}$, $v \in V$, write

$$X_l = \sum_{q \neq j} W^{(q)}(P_q h_l), \quad Y_l = W^{(j)}(P_j h_l), \quad l = 1, \dots, d,$$
$$X = (X_1, \dots, X_d), \quad Y = (Y_1, \dots, Y_d), \quad \bar{f}(x) = \mathbb{E}_j[f(x + Y)], \ x \in \mathbb{R}^d.$$

Polynomial growth of all derivatives gives

$$\partial^\alpha \bar{f} = \mathbb{E}_j[\partial^\alpha f(\cdot + Y)], \qquad \bar{f} \in C_p^\infty, \qquad \mathbb{E}_j F = \bar{f}(X)v \in \mathcal{S}(\mathcal{K}; V)$$

for any multiple index α. Since $\mathbf{k}_l \neq j$, differentiation commutes with $\mathbb{E}_j$ for every element of $\mathbf{k}$. Jensen's inequality gives $\|\mathbb{E}_j F\|_{\mathbf{k},p} \leq \|F\|_{\mathbf{k},p}$. For $F_m \to F$ in $\mathbb{D}^{k,p}(\Omega; V)$, closedness therefore yields

$$D^{\mathbf{k}}\mathbb{E}_j F = \lim_m D^{\mathbf{k}}\mathbb{E}_j F_m = \lim_m \mathbb{E}_j D^{\mathbf{k}} F_m = \mathbb{E}_j D^{\mathbf{k}} F.$$

□

Partial divergence operator. For $u \in L^2(\Omega; V \otimes \mathcal{H}_i)$, define

$$u \in \mathrm{Dom}\ \delta^{(i)} \iff \sup_{\substack{F \in \mathbb{D}^{1,2}(V) \\ \|F\|_{L^2(\Omega)} \leq 1}} |\mathbb{E}\langle D^{(i)}F, u\rangle_{V \otimes \mathcal{H}_i}| < \infty.$$

The Riesz representation theorem defines a unique $\delta^{(i)}u \in L^2(\Omega;V)$ by

$$\mathbb{E}\langle F, \delta^{(i)}u\rangle_V = \mathbb{E}\langle D^{(i)}F, u\rangle_{V\otimes\mathcal{H}_i} \quad (F \in \mathbb{D}^{1,2}(V)), \qquad \delta^{(i)} = (D^{(i)})^*.$$

Corollary 10. *Let $u \in$ Dom $\delta^{(i)}$. We have $u(\omega_{-i},\cdot) \in$ Dom δ_i and*

$$(\delta^{(i)}u)(\omega_{-i},\cdot) = \delta_i(u(\omega_{-i},\cdot)) \tag{5}$$

for $\mathbb{P}_{-i}$-a.s. ω_{-i}, where δ_i is a divergence operator on Ω_i with respect to $W^{(i)}$.

Proof. We have $(D^{(i)})^* = (I_{L^2(\Omega_{-i})}\widehat{\otimes}D_i)^* = I_{L^2(\Omega_{-i})}\widehat{\otimes}\delta_i$. Then

$$u \in \text{Dom } \delta^{(i)} \quad \Leftrightarrow \quad u \in L^2(\Omega_{-i};\text{Dom } \delta_i) \quad \Leftrightarrow \quad u(\omega_{-i},\cdot) \in \text{Dom } \delta_i, \quad \mathbb{P}_{-i}\text{−a.s.},$$
$$\int_{\Omega_{-i}} \|\delta_i(u(\omega_{-i},\cdot))\|^2_{L^2(\Omega_i;V)}\mathbb{P}_{-i}(d\omega_{-i}) < \infty.$$

The equation (5) holds for $u = u_{-i}u_i,\ u_{-i} \in L^2(\Omega_{-i}), u_i \in$ Dom δ_i. The general case follows from an approximation argument. □

Partial chaos decomposition. We define the projection to the partial r-th Wiener chaos of the i-th component by

$$\mathcal{J}_r^{(i)}(F) = (I_{L^2(\Omega_{-i})} \otimes \mathcal{J}_r)F = G_{-i}\mathcal{J}_r(G_i), \qquad G_{-i} \in L^2(\Omega_{-i}), \qquad G_i \in L^2(\Omega_i)$$

for $F = G_{-i}G_i$, where $\mathcal{J}_r$ is an ordinary projection to the rth chaos on Ω_i. For a general $F \in L^2(\Omega_{-i}\times\Omega_i)$, we define $\mathcal{J}_r^{(i)}$ by linearity and a limiting argument.

Lemma 11. *We have*

$$(\mathcal{J}_r^{(i)}F)(\omega_{-i},\cdot) = \mathcal{J}_r(F(\omega_{-i},\cdot))$$

for $\mathbb{P}_{-i}$-a.s. $\omega_{-i} \in \Omega_{-i}$.

Proof. An ordinary approximation argument gives

$$\int_{\Omega_{-i}} \mathbb{P}_{-i}(d\omega_{-i})\|(\mathcal{J}_r^{(i)}F)(\omega_{-i},\cdot) - \mathcal{J}_r(F(\omega_{-i},\cdot))\|^2_{L^2(\Omega_i)} = 0.$$

□

For a V-valued random variable $F \in L^2(\Omega;V)$, we have $F = \sum_{j=1}^\infty \langle F, e_j\rangle_V e_j$ and define

$$(I_V \otimes \mathcal{J}_r^{(i)})F = \sum_{j=1}^\infty \mathcal{J}_r^{(i)}\langle F, e_j\rangle_V e_j.$$

Partial Ornstein-Uhlenbeck semigroup. The Ornstein-Uhlenbeck semigroup is defined for $F \in L^2(\Omega;V)$ by

$$P_t^{(i)}F = \sum_{k=0}^\infty e^{-kt}(I_V \otimes \mathcal{J}_k^{(i)})F, \qquad t \geq 0.$$

Fix $1 \leq i < q \leq n$. Let $F \in L^2(\Omega;V)$ be $\sigma(W^{(1)},\dots,W^{(q-1)})$-measurable and let $\psi_F : \mathbb{R}^{\mathcal{H}_1} \times \cdots \times \mathbb{R}^{\mathcal{H}_{q-1}} \to V$ be a measurable mapping with $F = \psi_F(W^{(1)},\dots,W^{(q-1)})$, $\mathbb{P}$-a.s. Whenever there exists an isometry $U_{i,q} : \mathcal{H}_i \to \mathcal{H}_q$, we define

$$M_q^{(i)}(t)F = \psi_F(W^{(1)},\cdots,e^{-t}W^{(i)} + \sqrt{1-e^{-2t}}W^{(q)} \circ U_{i,q},\dots,W^{(q-1)}).$$

Remark. Later we will consider the case where $\mathcal{H}_1 = \ldots = \mathcal{H}_n$ and hence the existence of U will not be a problem.

The partial version of Mehler's formula holds.

Lemma 12. *Let $F \in L^2(\Omega; V)$ be measurable with respect to $\sigma(W^{(1)}, \ldots, W^{(q-1)})$. We have*

$$P_t^{(i)} F = \mathbb{E}_q\{M_q^{(i)}(t) F\} \tag{6}$$

for $i \in \{1, \ldots, q-1\}$.

Proof. First note that, for $\mathbb{P}_{-i}$-a.s. $\omega_{-i} \in \Omega_{-i}$, the random variable $\omega_i \mapsto F(\omega_{-i}, \omega_i)$ is in $L^2(\Omega_i; V)$. Then the ordinary Mehler formula gives

$$\begin{aligned}(P_t^{(i)} F)(\omega_{-i}, \cdot) = P_t(F(\omega_{-i}, \cdot)) &= \mathbb{E}_q \psi_F(W^{(1)}(\omega_1), \ldots, e^{-t} W^{(i)} + \sqrt{1 - e^{-2t}} W^{(q)} \circ U_{i,q}, \ldots, W^{(q-1)}(\omega_{q-1})) \\ &= \mathbb{E}_q\{(M_q^{(i)}(t) F)(\omega_{-i}, \cdot)\}.\end{aligned}$$

This equality holds for $\mathbb{P}_{-i}$-a.s. ω_{-i}, in the $L^2(\Omega_i)$ sense. This implies (6) in the $L^2(\Omega)$ sense. □

Partial Ornstein-Uhlenbeck operator. Note that $\mathcal{J}_0^{(i)} F = \mathbb{E}_i F$ and define

$$\text{Dom } L^{(i)} = \left\{ F \in L^2(\Omega) \;\middle|\; \sum_{r=0}^{\infty} r^2 \|\mathcal{J}_r^{(i)}(F)\|_{L^2(\Omega)}^2 < \infty \right\}, \qquad L^{(i)} F = -\sum_{r \geq 1} r \mathcal{J}_r^{(i)} F.$$

Using the ordinary Ornstein-Uhlenbeck operator L_i on Ω_i, we can rewrite $L^{(i)} = I_{L^2(\Omega_{-i})} \widehat{\otimes} L_i$ with the domain $L^2(\Omega_{-i}; \text{Dom } L_i)$.

Lemma 13. *For $F \in \text{Dom } L^{(i)}$,*

$$L^{(i)} F = -\delta^{(i)} D^{(i)} F.$$

Proof. This is because

$$L^{(i)} = I_{L^2(\Omega_{-i})} \widehat{\otimes} L_i = (I_{L^2(\Omega_{-i})} \widehat{\otimes} - \delta_i D_i) = -(I_{L^2(\Omega_{-i})} \widehat{\otimes} \delta_i)(I_{L^2(\Omega_{-i})} \widehat{\otimes} D_i) = -\delta^{(i)} D^{(i)}.$$

□

For $F \in L^2(\Omega; V)$, the pseudo-inverse is

$$(L^{(i)})^{-1} F = -\sum_{r \geq 1} \frac{1}{r} \mathcal{J}_r^{(i)} F, \qquad L^{(i)} (L^{(i)})^{-1} F = F - \mathbb{E}_i F.$$

Both assertions follow from orthogonality; in particular,

$$\sum_{r \geq 1} r^2 \|\mathcal{J}_r^{(i)} (L^{(i)})^{-1} F\|_2^2 = \|F - \mathbb{E}_i F\|_2^2.$$

The following are partial versions of well-known formulas. For the ordinary Malliavin operators and Ornstein–Uhlenbeck semigroup, see Nualart [2006, Chapter 1] and Nourdin and Peccati [2012, Chapter 2].

Proposition 14. *For $F, G \in \mathbb{D}^{(i),2}(\Omega; V)$, we have*

$$\mathbb{E}_i\{\langle F - \mathbb{E}_i F, G - \mathbb{E}_i G \rangle_V\} = \mathbb{E}_i \left\{ \left\langle -D^{(i)} (L^{(i)})^{-1} F, D^{(i)} G \right\rangle_{V \otimes \mathcal{H}_i} \right\}$$

$\mathbb{P}$-*a.s.*

Proof. Put $u = -D^{(i)}(L^{(i)})^{-1}F \in \operatorname{Dom}\delta^{(i)}$. Since $\delta^{(i)}u = F - \mathbb{E}_i F$, conditional duality yields

$$\begin{aligned}\mathbb{E}_i[\langle F - \mathbb{E}_i F, G - \mathbb{E}_i G\rangle_V] &= \mathbb{E}_i[\langle F - \mathbb{E}_i F, G\rangle_V] \\ &= \mathbb{E}_i[\langle G, \delta^{(i)}u\rangle_V] = \mathbb{E}_i\langle D^{(i)}G, u\rangle_{V\otimes\mathcal{H}_i}.\end{aligned}$$

Cauchy–Schwarz gives integrability of all products. □

Proposition 15. *Let $1 \le i < q \le n$. Let $U_{i,q} : \mathcal{H}_i \to \mathcal{H}_q$ be the isometry used in $M_q^{(i)}$. We assume $F \in \mathbb{D}^{(i),2}(\Omega; V)$ is measurable with respect to $\sigma(W^{(1)}, \dots, W^{(q-1)})$. Then the following statements hold.*

(1) We have $M_q^{(i)}(t)F \in \mathbb{D}^{(i),2}(\Omega; V) \cap \mathbb{D}^{(q),2}(\Omega; V)$ and

$$D^{(i)}M_q^{(i)}(t)F = e^{-t}M_q^{(i)}(t)(D^{(i)}F), \quad D^{(q)}M_q^{(i)}(t)F = \sqrt{1-e^{-2t}}(I_V \otimes U_{i,q})M_q^{(i)}(t)(D^{(i)}F).$$

(2) For $j \notin \{i, q\}$, if also $F \in \mathbb{D}^{(j),2}(\Omega; V)$,

$$D^{(j)}(M_q^{(i)}(t)F) = M_q^{(i)}(t)(D^{(j)}F).$$

(3) We have

$$-D^{(i)}(L^{(i)})^{-1}F = \int_0^\infty d\tau\ e^{-\tau}\mathbb{E}_q\{M_q^{(i)}(\tau)(D^{(i)}F)\} = \int_0^\infty d\tau\ \mathbb{E}_q\{D^{(i)}M_q^{(i)}(\tau)(F)\}$$

$\mathbb{P}$-*a.s.*

Proof. For $F \in \mathcal{S}(\Omega; V)$, (1) and (2) follow by the chain rule. Since $\mathcal{M}_q^{(i)}(t)$ is an L^2-isometry, smooth approximation and closedness of the derivatives extend (1) to $\mathbb{D}^{(i),2}(\Omega; V)$ and (2) to $\mathbb{D}^{(i),2}(\Omega; V) \cap \mathbb{D}^{(j),2}(\Omega; V)$. Since $D^{(i)}$ commutes with $\mathbb{E}_q$,

$$D^{(i)}P_t^{(i)}F = e^{-t}P_t^{(i)}D^{(i)}F.$$

For (3), set

$$Q_T = -\int_0^T (P_t^{(i)}F - \mathbb{E}_i F)\,dt = -\sum_{m\ge1}\frac{1-e^{-mT}}{m}\mathcal{J}_m^{(i)}F \longrightarrow (L^{(i)})^{-1}F \quad \text{in } L^2(\Omega; V).$$

The integral belongs to $\mathbb{D}^{(i),2}(\Omega; V)$, and

$$-D^{(i)}Q_T = \int_0^T e^{-t}P_t^{(i)}D^{(i)}F\,dt, \qquad \int_0^\infty e^{-t}\|P_t^{(i)}D^{(i)}F\|_{L^2(\Omega;V\otimes\mathcal{H}_i)}\,dt \le \|D^{(i)}F\|_{L^2(\Omega;V\otimes\mathcal{H}_i)}.$$

Hence we have (3). All integrals converge in the indicated Bochner L^2-spaces. □

A transfer formula for B^c. In section 3.2, we investigate the Malliavin derivative of the stationary solution $\bar{X}(B)$ of (2). However, in applications, we replace an fBM B with another fBM $B^c = \sum_i c_i B^{(i)}$ and consider the Malliavin derivative of $\bar{X}(B^c)$ with respect to $B^{(i)}$. In order to verify this calculation, we need a transfer theorem for a Malliavin derivative.

Let $\mathcal{H}'$ be another Hilbert space and $A : \mathcal{H}' \to \mathcal{K}$ be an isometric embedding. Let W' be isonormal over $\mathcal{H}'$ on a source space $(\Omega', \mathcal{F}', \mathbb{P}')$. The map A pulls $W^{\oplus}$ on $\mathcal{K}$ back to $\mathcal{H}'$ and this isonormal Gaussian process is denoted W^A: $W^A(h') = W^{\oplus}(Ah')$. Let $F' = f(W'(h'_1), \dots, W'(h'_m))v,\ f \in C_p^\infty(\mathbb{R}^m)$ and

$$F^A = f(W^A(h'_1), \dots, W^A(h'_m))v = f(W^{\oplus}(Ah'_1), \dots, W^{\oplus}(Ah'_m))v.$$

The Malliavin derivative of F^A with respect to $W^{(i)}$ is

$$D^{(i)}F^A = \sum_{l=1}^{m} (\partial_l f)(W^{\oplus}(Ah'_1), \dots, W^{\oplus}(Ah'_m))v \otimes P_i A h'_l = (I_V \otimes P_i A)\psi_{DF'}(W^A).$$

Here $\psi_{DF'}(w)$ is an equivalence class (with respect to $(\mathbb{P}')^{W'}$-a.s. identity) of functions whose representative is $\sum_{l=1}^{m} (\partial_l f)(w(h'_1), \dots, w(h'_m))v \otimes h'_l$. Higher order derivatives can be calculated for $\mathbf{k} = (\mathbf{k}_k, \dots, \mathbf{k}_1)$ and

$$\begin{aligned} D^{\mathbf{k}}F^A &= \sum_{l_1,\dots,l_k=1}^{m} \partial_{l_k} \cdots \partial_{l_1} f(W^{\oplus}(Ah'_1), \dots, W^{\oplus}(Ah'_m))v \otimes P_{\mathbf{k}_1} A h'_{l_1} \otimes \cdots \otimes P_{\mathbf{k}_k} A h'_{l_k} \\ &= (I_V \otimes P_{\mathbf{k}_1} A \otimes \cdots \otimes P_{\mathbf{k}_k} A)\psi_{D^k F'}(W^A). \end{aligned} \tag{7}$$

Here $I_V \otimes P_{\mathbf{k}_1} A \otimes \cdots \otimes P_{\mathbf{k}_k} A : V \otimes (\mathcal{H}')^{\otimes k} \to V \otimes \mathcal{H}_{\mathbf{k}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{k}_k}$ is

$$(I_V \otimes P_{\mathbf{k}_1} A \otimes \cdots \otimes P_{\mathbf{k}_k} A)(v \otimes h'_1 \otimes \cdots \otimes h'_k) = v \otimes P_{\mathbf{k}_1} A h'_1 \otimes \cdots \otimes P_{\mathbf{k}_k} A h'_k.$$

Assume that there are isometries $U_i : \mathcal{H}' \to \mathcal{H}_i$ for each $i = 1, \dots, n$. For $c = (c_1, \dots, c_n), |c|^2 = 1$, we set $A_c : \mathcal{H}' \to \mathcal{K}$ by $A_c h' = c_1 U_1 h' \oplus \dots \oplus c_n U_n h'$. Then A_c is an isometric embedding. Indeed

$$\|A_c h'\|_{\mathcal{K}}^2 = \sum_{j=1}^{n} c_j^2 \|U_j h'\|_{\mathcal{H}_j}^2 = \|h'\|_{\mathcal{H}'}^2 \sum_{j=1}^{n} c_j^2 = \|h'\|_{\mathcal{H}'}^2.$$

In this case $W^{A_c}(h) = \sum_{j=1}^{n} c_j W^{(j)}(U_j h)$ and

$$D^{\mathbf{k}}F^{A_c} = c_{\mathbf{k}_1} \dots c_{\mathbf{k}_k} (I_V \otimes U_{\mathbf{k}_1} \otimes \cdots \otimes U_{\mathbf{k}_k})\psi_{D^k F'}(W^{A_c}). \tag{8}$$

Furthermore, if $\mathcal{H}' = \mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$, then we can choose $U_i = \mathrm{Id}_{\mathcal{H}}$ and

$$D^{\mathbf{k}}F^{A_c} = c_{\mathbf{k}_1} \dots c_{\mathbf{k}_k} \psi_{D^k F'}(W^{A_c}). \tag{9}$$

For $F' = \psi_{F'}(W') \in \mathbb{D}^{k,p}(\Omega'; V)$, set $F^A = \psi_{F'}(W^A)$. The formulas (7), (8) and (9) extend by the following approximation argument.

Indeed, let $1 \le p < \infty$ and $F_m \in \mathcal{S}(\Omega'; V)$ converge to $F' \in \mathbb{D}^{k,p}(\Omega'; V)$. Since $W^A \overset{\text{law}}{=} W'$ and $\|P_i A\|_{\mathcal{H}_i} \le 1$,

$$\begin{aligned} \|F_m^A - F^A\|_{L^p(\Omega;V)} &= \|F_m - F'\|_{L^p(\Omega';V)} \longrightarrow 0, \\ \|D^{\mathbf{k}}(F_m^A - F_q^A)\|_{L^p(\Omega; V \otimes \mathcal{H}_{\mathbf{k}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{k}_l})} &\le \|D^\ell(F_m - F_q)\|_{L^p(\Omega'; V \otimes (\mathcal{H}')^{\otimes \ell})} \longrightarrow 0, \\ \mathbf{k} = (\mathbf{k}_l, \dots, \mathbf{k}_1), &\qquad 1 \le \ell \le k. \end{aligned}$$

Closedness passes the cylindrical formulas to the limit. Taking $A = A_c$ and $P_i A_c = c_i U_i$ proves the remaining assertions.

In this paper we use the last special case and hence summarize it as a proposition.

Proposition 16. *Let W be isonormal over $\mathcal{H}$ on $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, with independent copies $W^{(1)}, \dots, W^{(n)}$ on $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_0^n, \mathcal{F}_0^{\otimes n}, \mathbb{P}_0^{\otimes n})$. For $c = (c_1, \dots, c_n)$, $|c|^2 = 1$, an isometric embedding $A_c : \mathcal{H} \to \mathcal{K}$, $A_c(h) = c_1 h \oplus \cdots \oplus c_n h$ pulls $W^{\oplus}$ over $\mathcal{K}$ back to $\mathcal{H}$ and defines an isonormal Gaussian process $W^c = W^{\oplus} \circ A_c = \sum_{j=1}^n c_j W^{(j)}$ over $\mathcal{H}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.*

If $F = \psi_F(W) \in \mathbb{D}^{k,p}(\Omega_0; V)$ and $D^k F = \psi_{D^k F}(W)$, then for any $\mathbf{k} = (\mathbf{k}_k, \dots, \mathbf{k}_1)$, we have $F^c = \psi_F(W^c) \in \mathbb{D}^{\mathbf{k},p}(\Omega; V)$ and

$$D^{\mathbf{k}} F^c = c_{\mathbf{k}_k} \cdots c_{\mathbf{k}_1} (\psi_{D^k F} \circ W^c).$$

3.2 The stationary solution

We consider applying Malliavin calculus to the stationary solution $\bar{X}_t^\theta$. This is essentially done in Chiba [2018]. However we need the Malliavin differentiability up to order three in this paper. This is given in this section.

The space of continuous functions ω with $\omega(0) = 0$ is denoted by $\Omega_0 = C_0(\mathbb{R})$. We define $\mathcal{F}_0$ as a completion of the σ-algebra that is induced by the topology of compact convergence. There exists a probability measure $\mathbb{P}_0$ such that the canonical process $B_t(\omega) = \omega(t)$ is a two-sided fractional Brownian motion. For $\omega \in \Omega_0$, we define a shift operator ϑ_s as $(\vartheta_s \omega)(t) = \omega(t+s) - \omega(s)$. It is known that there exists a measurable set $\Omega_{\mathrm{good}} \subset \Omega_0$ with $\mathbb{P}_0(\Omega_{\mathrm{good}}) = 1$ and $\vartheta_s \Omega_{\mathrm{good}} = \Omega_{\mathrm{good}}$ such that if $\omega \in \Omega_{\mathrm{good}}$ then

$$|\omega(t)| \le C(\omega)(1 + |t|^2) \quad \forall t \in \mathbb{R}$$

for some $C(\omega) > 0$; see Chiba [2018, Section 3, p. 16].

We construct the stationary solution on Ω_{good}. We set

$$X_t^{x,\theta}(\omega) = x + \int_0^t a(X_s^{x,\theta}(\omega), \theta) ds + \sigma B_t(\omega)$$

for all $\omega \in \Omega_0$.

Proposition 17. *For each $\omega \in \Omega_{\mathrm{good}}$, there exists $\lim_{n\to\infty} X_n^{x,\theta}(\vartheta_{-n}\omega)$ that is independent of $x \in \mathbb{R}$.*

Proof. First note that $(t, x, \omega) \mapsto X_t^{x,\theta}(\omega)$ has the cocycle property: $X_\cdot^{X_s^{x,\theta}(\omega),\theta}(\vartheta_s \omega) = X_{\cdot+s}^{x,\theta}(\omega)$ holds for any $s \in \mathbb{R}$ and $\omega \in \Omega_0$.

We have

$$X_t^{x,\theta}(\omega) - X_t^{y,\theta}(\omega) = x - y + \int_0^t (a(X_s^{x,\theta}, \theta) - a(X_s^{y,\theta}, \theta)) ds$$

and the assumption (A1) and Gronwall's inequality give

$$|X_t^{x,\theta}(\omega) - X_t^{y,\theta}(\omega)| \le e^{-\alpha t} |x - y|. \tag{10}$$

On the other hand, the cocycle property gives $X_{n+1}^{x,\theta}(\vartheta_{-(n+1)}\omega) = X_n^{X_1^{x,\theta}(\vartheta_{-(n+1)}\omega),\theta}(\vartheta_{-n}\omega)$. Hence combining with (10) we have

$$|X_{n+1}^{x,\theta}(\vartheta_{-(n+1)}\omega) - X_n^{x,\theta}(\vartheta_{-n}\omega)| \le e^{-\alpha n} |X_1^{x,\theta}(\vartheta_{-(n+1)}\omega) - x|. \tag{11}$$

Since

$$\begin{aligned} X_1^{x,\theta}(\vartheta_{-(n+1)}\omega) - x = \int_0^1 dr \int_0^1 d\epsilon (\partial_x a)(\epsilon X_r^{x,\theta}(\vartheta_{-(n+1)}\omega) + (1-\epsilon)x, \theta)(X_r^{x,\theta}(\vartheta_{-(n+1)}\omega) - x) \\ + a(x,\theta) + \sigma(\omega(-n) - \omega(-n-1)), \end{aligned}$$

again the assumption (A1) and Gronwall's inequality give

$$|X_1^{x,\theta}(\vartheta_{-(n+1)}\omega) - x| \le C(x,\theta,\omega,\sigma,\alpha,\alpha')(1+n^2)$$

for each $\omega \in \Omega_{\text{good}}$. This implies $(X_n^{x,\theta}(\vartheta_{-n}\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence. The inequality (10) also shows the limit is independent of x. □

We set $\bar{X}_0^\theta(\omega) = \lim_{n\to\infty} X_n^{x,\theta}(\vartheta_{-n}\omega)$ and $\bar{X}_t^\theta(\omega) = \bar{X}_0^\theta(\vartheta_t\omega)$.

Remark. Note that $\lim_{T\to\infty} X_T^{x,\theta}(\vartheta_{-T}\omega) = \bar{X}_0(\omega)$ for each $\omega \in \Omega_{\text{good}}$. In fact, the cocycle property gives $X_T^{x,\theta}(\vartheta_{-T}\omega) = X_{\lfloor T\rfloor}^{X_{T-\lfloor T\rfloor}^{x,\theta}(\vartheta_{-T}\omega),\theta}(\vartheta_{-\lfloor T\rfloor}\omega)$. Hence

$$|X_T^{x,\theta}(\vartheta_{-T}\omega) - X_{\lfloor T\rfloor}^{x,\theta}(\vartheta_{-\lfloor T\rfloor}\omega)| \le e^{-\alpha\lfloor T\rfloor}|X_{T-\lfloor T\rfloor}^{x,\theta}(\vartheta_{-T}\omega) - x|.$$

A calculation similar to that in Proposition 17 shows that $|X_{T-\lfloor T\rfloor}^{x,\theta}(\vartheta_{-T}\omega) - x|$ is of polynomial growth with respect to T.

This limit is also $L^p(\Omega)$ for any $p \ge 1$. This is because $\sup_{T>0} E\{|X_T^{x,\theta}(\vartheta_{-T})|^q\} < \infty$ for any $q \ge 1$ (see Chiba, 2018, Proposition 3.2).

Proposition 18. *The process $\bar{X}_t^\theta$ is the unique stationary process satisfying*

$$\bar{X}_t^\theta - \bar{X}_s^\theta = \int_s^t a(\bar{X}_r,\theta)\,dr + \sigma(B_t - B_s) \tag{12}$$

for any $s < t$, $\mathbb{P}$-a.s.

Proof. A proof for the equation (12) is found in Chiba [2018, Proposition 3.5]. Let $\bar{Y}^\theta$ be another stationary solution satisfying (12). Since $(X_{T+t}^{\bar{Y}_{-T},\theta}(\vartheta_{-T}\omega))_{t\in[-T,0]}$ and $(\bar{Y}_t^\theta(\omega))_{t\in[-T,0]}$ solve the same equation, $X_T^{\bar{Y}_{-T}(\omega),\theta}(\vartheta_{-T}\omega) = \bar{Y}_0^\theta(\omega)$. Hence

$$|\bar{Y}_0^\theta(\omega) - X_T^{x,\theta}(\vartheta_{-T}\omega)| \le e^{-\alpha T}|\bar{Y}_{-T}^\theta(\omega) - x|.$$

By stationarity of $\bar{Y}^\theta$, we have $\lim_{T\to\infty} X_T^{x,\theta}(\vartheta_{-T}) = \bar{Y}_0^\theta$ in probability. On the other hand we have $\lim_{T\to\infty} X_T^{x,\theta}(\vartheta_{-T}) = \bar{X}_0^\theta$, $\mathbb{P}$-a.s. This implies $\bar{X}_0^\theta = \bar{Y}_0^\theta$, $\mathbb{P}$-a.s. and after taking a countable intersection we have $\bar{X}^\theta = \bar{Y}^\theta$, $\mathbb{P}$-a.s. □

Proposition 19. *We set $J_{t,s}^\theta = \exp\left(\int_s^t (\partial_x a)(\bar{X}_r^\theta,\theta)dr\right)\mathbf{1}_{(-\infty,t]}(s)$ for $s,t \in \mathbb{R}$.*

(1) *For any $t \in \mathbb{R}$, the random variable $\bar{X}_t^\theta$ has Malliavin derivatives up to order three, and its Malliavin derivatives are given by*

$$D_{s_1}\bar{X}_t^\theta = \sigma J_{t,s_1}^\theta, \tag{13}$$

$$D_{s_2}D_{s_1}\bar{X}_t^\theta = \int_{-\infty}^t (\partial_x^2 a)(\bar{X}_r^\theta,\theta)J_{t,r}^\theta (D_{s_1}\bar{X}_r^\theta)(D_{s_2}\bar{X}_r^\theta)\,dr, \tag{14}$$

and

$$\begin{aligned} D_{s_3}D_{s_2}D_{s_1}\bar{X}_t^\theta = \int_{-\infty}^t J_{t,r}^\theta \Big[& \partial_x^3 a(\bar{X}_r^\theta,\theta)(D_{s_1}\bar{X}_r^\theta)(D_{s_2}\bar{X}_r^\theta)(D_{s_3}\bar{X}_r^\theta) \\ & + \partial_x^2 a(\bar{X}_r^\theta,\theta)\Big\{(D_{s_3}D_{s_1}\bar{X}_r^\theta)(D_{s_2}\bar{X}_r^\theta) + (D_{s_3}D_{s_2}\bar{X}_r^\theta)(D_{s_1}\bar{X}_r^\theta) \\ & + (D_{s_1}D_{s_2}\bar{X}_r^\theta)(D_{s_3}\bar{X}_r^\theta)\Big\}\Big]dr. \end{aligned} \tag{15}$$

(2) *Furthermore, these Malliavin derivatives satisfy the inequalities* $\sup_{t\in\mathbb{R}} \|D\bar{X}_t^\theta\|_{\mathcal{H}} < C(\sigma, H, \alpha)$, $\sup_{t\in\mathbb{R}} \|D^2\bar{X}_t^\theta\|_{\mathcal{H}^{\otimes 2}} < C(\sigma, H, \alpha, \|\partial_x^2 a\|_\infty)$, *and* $\sup_{t\in\mathbb{R}} \|D^3\bar{X}_t^\theta\|_{\mathcal{H}^{\otimes 3}} < C(\sigma, H, \alpha, \|\partial_x^2 a\|_\infty, \|\partial_x^3 a\|_\infty)$, $\mathbb{P}$*-a.s. In particular* $\bar{X}_t^\theta \in \mathbb{D}^{3,p}$ *for all* $p \geq 1$.

Proof. In order to investigate the Malliavin differentiability of $\bar{X}_t^\theta$, we set $Z_t^{x,\theta,T}(\omega) = X_{t+T}^{x,\theta}(\vartheta_{-T}\omega)$, $t \geq -T$. The process $Z_t^{x,\theta,T}$ solves an SDE

$$Z_t^{x,\theta,T} = x + \int_{-T}^t a(Z_r^{x,\theta,T}, \theta)dr + \sigma(B_t - B_{-T}), \quad t \geq -T. \tag{16}$$

Let $J_{t,s}^{\theta,T} = \exp\left(\int_s^t (\partial_x a)(Z_r^{x,\theta,T}, \theta)dr\right)\mathbf{1}_{[-T,t]}(s)$. The Malliavin differentiability (up to order three) of $Z_t^{x,\theta,T}$ can be shown by an ordinary Picard approximation argument. Its explicit formulas up to order three are given by

$$\begin{aligned}
D_{s_1} Z_t^{x,\theta,T} &= \sigma J_{t,s_1}^{\theta,T}, \\
D_{s_2} D_{s_1} Z_t^{x,\theta,T} &= \int_{-T}^t (\partial_x^2 a)(Z_r^{x,\theta,T}, \theta) J_{t,r}^{\theta,T} (D_{s_1} Z_r^{x,\theta,T})(D_{s_2} Z_r^{x,\theta,T})\, dr,
\end{aligned}$$

and

$$\begin{aligned}
D_{s_3} D_{s_2} D_{s_1} Z_t^{x,\theta,T} = \int_{-T}^t J_{t,r}^{\theta,T} \Big[& \partial_x^3 a(Z_r^{x,\theta,T}, \theta)(D_{s_1} Z_r^{x,\theta,T})(D_{s_2} Z_r^{x,\theta,T})(D_{s_3} Z_r^{x,\theta,T}) \\
& + \partial_x^2 a(Z_r^{x,\theta,T}, \theta) \Big[(D_{s_3} D_{s_1} Z_r^{x,\theta,T})(D_{s_2} Z_r^{x,\theta,T}) + (D_{s_3} D_{s_2} Z_r^{x,\theta,T})(D_{s_1} Z_r^{x,\theta,T}) \\
& + (D_{s_1} D_{s_2} Z_r^{x,\theta,T})(D_{s_3} Z_r^{x,\theta,T}) \Big] \Big] dr.
\end{aligned}$$

These formulas are obtained by solving linear equations

$$\begin{aligned}
DZ_t^{x,\theta,T} &= \int_{-T}^t (\partial_x a)(Z_r^{x,\theta,T}, \theta) DZ_r^{x,\theta,T} dr + \sigma \mathbf{1}_{[-T,t]}, \\
D^2 Z_t^{x,\theta,T} &= \int_{-T}^t \left[(\partial_x a)(Z_r^{x,\theta,T}, \theta)(D^2 Z_r^{x,\theta,T}) + (\partial_x^2 a)(Z_r^{x,\theta,T}, \theta)(DZ_r^{x,\theta,T})^{\otimes 2} \right] dr,
\end{aligned}$$

and

$$\begin{aligned}
D^3 Z_t^{x,\theta,T} = \int_{-T}^t \Big[& (\partial_x a)(Z_r^{x,\theta,T}, \theta)(D^3 Z_r^{x,\theta,T}) + (\partial_x^3 a)(Z_r^{x,\theta,T}, \theta)(DZ_r^{x,\theta,T})^{\otimes 3} \\
& + (\partial_x^2 a)(Z_r^{x,\theta,T}, \theta) 3\mathrm{Sym}_3((D^2 Z_r^{x,\theta,T}) \otimes (DZ_r^{x,\theta,T})) \Big] dr.
\end{aligned}$$

Here $\mathrm{Sym}_3 : \mathcal{H}^{\otimes 3} \to \mathcal{H}^{\odot 3}$ is a symmetrization operator

$$\mathrm{Sym}_3(h_1 \otimes h_2 \otimes h_3) = \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes h_{\sigma(3)}.$$

By Chiba [2018, Lemmas 3.7 and 3.10], we have $\sup_{T>0} \sup_{t\geq -T} \|DZ_t^{x,\theta,T}\|_{\mathcal{H}} \leq C(\sigma, H, \alpha, \alpha') =: C_1$,

$\mathbb{P}$-a.s. A bound for $\|D^2 Z_t^{x,\theta,T}\|_{\mathcal{H}^{\otimes 2}}$ is

$$\begin{aligned}\|D^2 Z_t^{x,\theta,T}\|_{\mathcal{H}^{\otimes 2}} &\le \|\partial_x^2 a\|_\infty \int_{-T}^t dr\ |J_{t,r}^{\theta,T}| \|DZ_r^{x,\theta,T}\|_{\mathcal{H}}^2 \\ &\le \|\partial_x^2 a\|_\infty C_1^2 \int_{-\infty}^t dr\ e^{-\alpha(t-r)} \\ &= \frac{\|\partial_x^2 a\|_\infty C_1^2}{\alpha} \\ &=: C_2.\end{aligned}$$

This bound is independent of $t \ge -T$, $T > 0$ and $\omega \in \Omega_0$. Hence $\sup_{T>0} \sup_{t\ge -T} \|D^2 Z_t^{x,\theta,T}\|_{\mathcal{H}^{\otimes 2}} < \infty$, $\mathbb{P}$-a.s. Similarly, a bound for $\|D^3 Z_t^{x,\theta,T}\|_{\mathcal{H}^{\otimes 3}}$ is

$$\begin{aligned}\|D^3 Z_t^{x,\theta,T}\|_{\mathcal{H}^{\otimes 3}} &\le \int_{-T}^t dr\ e^{-\alpha(t-r)} (\|\partial_x^3 a\|_\infty \|DZ_r^{x,\theta,T}\|_{\mathcal{H}}^3 + 3\|\partial_x^2 a\|_\infty \|D^2 Z_r^{x,\theta,T}\|_{\mathcal{H}^{\otimes 2}} \|DZ_r^{x,\theta,T}\|_{\mathcal{H}}) \\ &\le \frac{\|\partial_x^3 a\|_\infty C_1^3 + 3\|\partial_x^2 a\|_\infty C_2 C_1}{\alpha} \\ &=: C_3\end{aligned}$$

and the right hand side is a constant independent of $t \ge -T$, $T > 0$ and $\omega \in \Omega_0$.

By formally taking the limit $T \to \infty$, we expect that the right hand sides of (13), (14) and (15) are candidates for $D\bar{X}_t^\theta, D^2\bar{X}_t^\theta$ and $D^3\bar{X}_t^\theta$. They are denoted by $G_t^{(1)}, G_t^{(2)}$ and $G_t^{(3)}$. Note that the same calculation as above gives $\sup_{t\in\mathbb{R}} \|G_t^{(i)}\|_{\mathcal{H}^{\otimes i}} < \infty,\ i = 1, 2, 3$, $\mathbb{P}-$a.s.

First we show $DZ_t^{x,\theta,T} \to G_t^{(1)}$ in $L^p(\Omega; \mathcal{H})$. Note that since $Z_t^{x,\theta,T}$ and $\bar{X}^\theta$ are driven by the same noise, an ordinary Gronwall statement gives

$$|Z_r^{x,\theta,T} - \bar{X}_r^\theta| \le e^{-\alpha(r+T)} |x - \bar{X}_{-T}^\theta|.$$

Chiba [2018, Proposition 3.8] shows the Malliavin differentiability of $\bar{X}_t^\theta$ and

$$DZ_t^{x,\theta,T} = \int_{-T}^t (\partial_x a)(Z_r^{x,\theta,T}, \theta) DZ_r^{x,\theta,T} dr + \sigma \mathbf{1}_{(-T,t]}, \tag{17}$$

$$D\bar{X}_t^\theta = D\bar{X}_{-T}^\theta + \int_{-T}^t (\partial_x a)(\bar{X}_r^\theta, \theta) D\bar{X}_r^\theta dr + \sigma \mathbf{1}_{(-T,t]} \tag{18}$$

We set $U_t^T = DZ_t^{x,\theta,T} - D\bar{X}_t^\theta$. Then

$$U_t^T = -D\bar{X}_{-T}^\theta + \int_{-T}^t (\partial_x a)(Z_r^{x,\theta,T}, \theta) U_r^T dr + \int_{-T}^t ((\partial_x a)(Z_r^{x,\theta,T}, \theta) - (\partial_x a)(\bar{X}_r^\theta, \theta)) D\bar{X}_r dr. \tag{19}$$

Solving (19), we have

$$U_t^T = -J_{t,-T}^T D\bar{X}_{-T}^\theta + \int_{-T}^t J_{t,r}^T ((\partial_x a)(Z_r^{x,\theta,T}, \theta) - (\partial_x a)(\bar{X}_r^\theta, \theta)) D\bar{X}_r dr.$$

The norm of U_t^T becomes

$$\begin{aligned}\|U_t^T\|_{\mathcal{H}} &\le e^{-\alpha(t+T)}\|D\bar{X}_{-T}^{\theta}\|_{\mathcal{H}} + \|\partial_x^2 a\|_\infty \int_{-T}^{t} e^{-\alpha(t-r)}|Z_r^{x,\theta,T} - \bar{X}_r^{\theta}|\|D\bar{X}_r^{\theta}\|_{\mathcal{H}} dr \\ &\le C_1 e^{-\alpha(t+T)} + \|\partial_x^2 a\|_\infty C_1 |x - \bar{X}_{-T}^{\theta}| \int_{-T}^{t} e^{-\alpha(t-r)} e^{-\alpha(r+T)}\, dr \\ &= C_1 e^{-\alpha(t+T)} + \|\partial_x^2 a\|_\infty C_1 |x - \bar{X}_{-T}^{\theta}| e^{-\alpha(t+T)}(t+T).\end{aligned}$$

By the stationarity of $\bar{X}^\theta$, we conclude $U_t^T \to 0$ in $L^p(\Omega;\mathcal{H})$.

If we write $D^2 Z_t^{x,\theta,T} = \int_{-\infty}^t Q_{t,r}^{x,\theta,T} dr$ and $G_t^{(2)} = \int_{-\infty}^t Q_{t,r}^{\theta} dr$, then $\|Q_{t,r}^{x,\theta,T} - Q_{t,r}^{\theta}\|_{L^p(\Omega;\mathcal{H}^{\otimes 2})} \to 0$ as $T \to \infty$ thanks to $\|DZ_t^{x,\theta,T} - G_t^{(1)}\|_{L^p(\Omega;\mathcal{H})} \to 0$ $(T \to \infty)$. A uniform bound

$$\sup_{T>0} \|Q_{t,r}^{x,\theta,T}\|_{\mathcal{H}^{\otimes 2}} + \|Q_{t,r}^{\theta}\|_{\mathcal{H}^{\otimes 2}} \le \|\partial_x^2 a\|_\infty C(\sigma, H, \alpha)^2 e^{-\alpha(t-r)}$$

allows us to apply the dominated convergence theorem, yielding

$$\|D^2 Z_t^{x,\theta,T} - G_t^{(2)}\|_{L^p(\Omega;\mathcal{H}^{\otimes 2})} \le \int_{-\infty}^t \|Q_{t,r}^{x,\theta,T} - Q_{t,r}^{\theta}\|_{L^p(\Omega;\mathcal{H}^{\otimes 2})} dr \to 0$$

as $T \to \infty$. The limit $\|D^3 Z_t^{x,\theta,T} - G_t^{(3)}\|_{L^p(\Omega;\mathcal{H}^{\otimes 3})} \to 0$ as $T \to \infty$ is shown by an analogous argument. We conclude that $\bar{X}_t^\theta \in \mathbb{D}^{3,p}$ and $D^j \bar{X}_t^\theta = G_t^{(j)}$, $j = 1, 2, 3$.

□

Proof of Theorem 3. **Construction.** We set $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_0^n, \mathcal{F}_0^{\otimes n}, \mathbb{P}_0^{\otimes n})$ and $B^{(i)}(\omega_1, \dots, \omega_n) = \omega_i$. Then $B^{(1)}, \dots, B^{(n)}$ are independent fBMs on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Phi_c : \Omega \to \Omega_0$ be $\Phi_c(\omega_1, \dots, \omega_n) = \sum_{j=1}^n c_j \omega_j$. Then $B_t \circ \Phi_c = \sum_{j=1}^n c_j B_t^{(j)}$ is again a two-sided fBM. The process $Y^{c,\theta}$ is defined by $Y^{c,\theta} = \bar{X}^\theta \circ \Phi_c$.

Stationarity and uniqueness. Since $\mathbb{P}^{\Phi_c} = \mathbb{P}_0$ and $\bar{X}^\theta$ is stationary, the process $Y^{c,\theta}$ is also stationary. If $\tilde{Y}^{c,\theta}$ is another stationary process satisfying (3), then $Y^{c,\theta}$ and $\tilde{Y}^{c,\theta}$ satisfy the same equation

$$Z_t^{c,\theta} = Z_{-T}^{c,\theta} + \int_{-T}^{t} a(Z_r^{c,\theta}, \theta) dr + \sigma(B_t(\Phi_c) - B_{-T}(\Phi_c)), \quad (-T < t).$$

Hence $|Y_t^{c,\theta} - \tilde{Y}_t^{c,\theta}| \le e^{-\alpha(t+T)} |Y_{-T}^{c,\theta} - \tilde{Y}_{-T}^{c,\theta}|$. On the other hand,

$$\sup_{T>0} \mathbb{P}\left\{|Y_{-T}^{c,\theta} - \tilde{Y}_{-T}^{c,\theta}| > R\right\} \le \mathbb{P}\left\{|Y_0^{c,\theta}| > R/2\right\} + \mathbb{P}\left\{|\tilde{Y}_0^{c,\theta}| > R/2\right\} \to 0 \quad (R \to \infty)$$

and we have $\mathbb{P}\left\{|Y_t^{c,\theta} - \tilde{Y}_t^{c,\theta}| > \epsilon\right\} = 0$ for any $\epsilon > 0$. After taking a countable intersection of sets of full $\mathbb{P}$-measure, we can conclude $Y^{c,\theta} = \tilde{Y}^{c,\theta}$.

Malliavin derivatives. The processes $B, B^{(1)}, \dots, B^{(n)}$ define isonormal Gaussian processes over $\mathcal{H} = I_-^{1/2-H}(L^2(\mathbb{R}))$. The composition $B(h) \circ \Phi_c = \sum_{j=1}^n c_j B^{(j)}(h)$ is also an isonormal Gaussian process over $\mathcal{H}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. This coincides with $W^c(h) = W^{\oplus}(A_c h)$ in Proposition 16 (in this proof we denote $B^c = B^{\oplus}(A_c h)$). We take measurable representatives $\psi_t^\theta, \psi_{t,r}^\theta$ $(r = 1, 2, 3)$ with $\bar{X}_t^\theta = \psi_t^\theta(B)$ and $D^r \bar{X}_t^\theta = \psi_{t,r}^\theta(B)$. Then

$$\psi_t^\theta(B^c) = \psi_t^\theta \circ B \circ \Phi_c = \bar{X}_t^\theta \circ \Phi_c = Y_t^{c,\theta},$$

$\mathbb{P}$-a.s. Hence by Proposition 16, we have

$$D^{\mathbf{k}} Y_t^{c,\theta} = c_{\mathbf{k}_1} \cdots c_{\mathbf{k}_k} \psi_{t,k}^\theta(B^c)$$

for a multi-index $\mathbf{k} = (\mathbf{k}_k, \dots, \mathbf{k}_1)$. By Proposition 19,

$$D^{(\mathbf{k}_1)} Y_t^{c,\theta} = c_{\mathbf{k}_1} \psi_{t,1}^{\theta}(B \circ \Phi_c) = c_{\mathbf{k}_1} \sigma \exp\left(\int_{\cdot}^{t} (\partial_x a)(\bar{X}_r^{\theta} \circ \Phi_c, \theta) dr \right) \mathbf{1}_{(-\infty,t]}(\cdot) = \sigma c_{\mathbf{k}_1} J_{t,\cdot}^{c,\theta}.$$

The higher order derivatives are calculated in the same way.

□

4 Cumulant Calculation

We use standard graph-theoretic terminology for trees, forests, degrees, and leaves in Sections 4.2–4.3; see Diestel [2017, Chapter 1].

4.1 A representation for the joint fourth cumulant

A joint fourth cumulant. For real random variables $F_0, \dots, F_3 \in L^4(\Omega)$, put $\widetilde{F}_v = F_v - \mathbb{E}F_v$. Their joint fourth cumulant is

$$\begin{aligned} \kappa_4(F_0, F_1, F_2, F_3) = \mathbb{E}\{\widetilde{F}_0 \widetilde{F}_1 \widetilde{F}_2 \widetilde{F}_3\} - \mathbb{E}\{\widetilde{F}_0 \widetilde{F}_1\} \mathbb{E}\{\widetilde{F}_2 \widetilde{F}_3\} \\ - \mathbb{E}\{\widetilde{F}_0 \widetilde{F}_2\} \mathbb{E}\{\widetilde{F}_1 \widetilde{F}_3\} - \mathbb{E}\{\widetilde{F}_0 \widetilde{F}_3\} \mathbb{E}\{\widetilde{F}_1 \widetilde{F}_2\}. \end{aligned} \tag{20}$$

The Γ-operator. On the product Gaussian space of Section 3, define

$$A_i U = -D^{(i)}(L^{(i)})^{-1} U, \qquad \Gamma_i(U, V) = \langle A_i U, D^{(i)} V \rangle_{\mathcal{H}_i}.$$

The partial integration-by-parts formula reads

$$\mathbb{E}_i\{(U - \mathbb{E}_i U)V\} = \mathbb{E}_i\{\Gamma_i(U, V)\}. \tag{21}$$

For $U, V \in \mathbb{D}^{(i),2}$ this is an L^1 identity in the other Gaussian components. When there is only one Gaussian component, we write $A = -DL^{-1}$ and $\Gamma(U, V) = \langle AU, DV \rangle_{\mathcal{H}}$.

Lemma 20 (Iterated Gamma identity)**.** *Suppose that $F_0, \dots, F_3 \in \bigcap_{1<p<\infty} \mathbb{D}^{3,p}$ are centred functionals of one isonormal Gaussian process. Write S_3 for the six permutations (i, j, k) of $(1, 2, 3)$. Then*

$$\kappa_4(F_0, F_1, F_2, F_3) = \sum_{(i,j,k) \in S_3} \mathbb{E}\{\Gamma(\Gamma(\Gamma(F_0, F_i), F_j), F_k)\}. \tag{22}$$

Proof. Set $G_i = \Gamma(F_0, F_i)$, so that $\mathbb{E}G_i = \mathbb{E}(F_0 F_i)$. Integration by parts and the product rule give

$$\kappa_4(F_0, F_1, F_2, F_3) = \sum_{i=1}^{3} \mathbb{E}\{(G_i - \mathbb{E}G_i) F_j F_k\}, \qquad \{j, k\} = \{1, 2, 3\} \setminus \{i\}.$$

For each i the summand equals

$$\mathbb{E}\{\Gamma(G_i, F_j) F_k\} + \mathbb{E}\{\Gamma(G_i, F_k) F_j\}.$$

Since F_j, F_k are centred, one more integration by parts gives (22). □

Remark. Rewriting cumulants using an iterated gamma operator is, for example, seen in Nourdin and Peccati [2010] where $F_1 = F_2 = F_3 = F_4$.

An interpolation with independent noise. Let $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ be an underlying probability space and let $W^{(0)}$ be an isonormal Gaussian process. The independent isonormal processes $W^{(q)}$, $q = 0, 1, 2, 3$, over $\mathcal{H}$ are defined on the product probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_0^4, \mathcal{F}_0^4, \mathbb{P}_0^{\otimes 4})$ by

$$W^{(q)} = W^{(0)} \circ \Pi_q, \qquad \Pi_q(\omega_0, \omega_1, \omega_2, \omega_3) = \omega_q.$$

Every functional F_v of $W^{(0)}$ on Ω_0 is identified with $F_v = \psi_v(W^{(0)})$ on Ω. We set $\mu = \mathbb{P} \circ (W^{(0)})^{-1}$ and then ψ_v is determined μ-a.s. On such functionals, the ordinary Malliavin operators coincide with their partial counterparts in the 0th coordinate: $\psi_{DF} = \psi_{D^{(0)}F}$ holds μ-a.s. In particular,

$$\Gamma(U, V) = \Gamma_0(U, V) := \langle A_0 U, D^{(0)} V \rangle_{\mathcal{H}}, \qquad A_0 = -D^{(0)}(L^{(0)})^{-1},$$

and hence

$$\mathbb{E}_0\{\Gamma(\Gamma(\Gamma(F_0, F_i), F_j), F_k)\} = \mathbb{E}_0\{\Gamma_0(\Gamma_0(\Gamma_0(F_0, F_i), F_j), F_k)\}.$$

For $q = 1, 2, 3$, put $a_q = e^{-\tau_q}, b_q = (1 - a_q^2)^{1/2}$, and let M_q be the partial Mehler substitution

$$M_q(\psi_F(W^{(0)}, W^{(1)}, \ldots, W^{(q-1)})) = \psi_F(a_q W^{(0)} + b_q W^{(q)}, W^{(1)}, \ldots, W^{(q-1)})$$

for measurable $\psi_F : (\mathbb{R}^{\mathcal{H}})^q \to \mathbb{R}$. By the partial Mehler formula, A_0 in the qth gamma operator becomes

$$A_0 U = \int_0^\infty a_q \mathbb{E}_q\{M_q D^{(0)} U\}\, d\tau_q.$$

For $c = (c_0, c_1, c_2, c_3) \in \mathbb{R}^4$, write $W^c = \sum_{r=0}^3 c_r W^{(r)}$. Whenever $c_q = 0$, the substitution M_q transforms W^c into $W^{R_q c}$, where

$$R_q c = a_q c_0 e_0 + b_q c_0 e_q + \sum_{r \neq 0, q} c_r e_r.$$

In particular, R_q preserves the norm of such coefficient vectors.

For a permutation (i, j, k) of $(1, 2, 3)$, the four noises occurring after the three successive Mehler substitutions are therefore

$$c_0 = R_3 R_2 R_1 e_0, \qquad c_i = R_3 R_2 e_0, \qquad c_j = R_3 e_0, \qquad c_k = e_0,$$

that is,

$$\begin{aligned} c_0 &= (a_1 a_2 a_3,\ b_1,\ a_1 b_2,\ a_1 a_2 b_3), \\ c_i &= (a_2 a_3,\ 0,\ b_2,\ a_2 b_3), \\ c_j &= (a_3,\ 0,\ 0,\ b_3), \\ c_k &= (1,\ 0,\ 0,\ 0). \end{aligned}$$

For $v \in \{0, i, j, k\}$ and $1 \leq r \leq 3$, choose a measurable representative $D^r F_v = \psi_{D^r F_v}(W)$ and set

$$F_v^{[r], c} := \psi_{D^r F_v}(W^c) \in \mathcal{H}^{\otimes r}.$$

Contraction operator. Let $h \otimes_r g$ denote the r-th contraction of tensors (see Nourdin and Peccati, 2012, Appendix B, Section B.4).

Definition 21 (Contraction operators)**.** Let $T = (V, E)$ be a tree and set

$$V = \{0, i, j, k\}, \quad E(T) \subset \{\{v, w\} : v, w \in V,\ v \neq w\}, \quad E_T(v) = \{e \in E(T) : v \in e\}, \quad d_T(v) = \#E_T(v).$$

Let $U_v \in \mathcal{H}^{\otimes d_T(v)}$ be symmetric tensors. We define an iterated 1-contraction operator $\mathcal{C}_T$ by the following procedure. Choose a vertex v with $d_T(v) = 1$ and let $\{v, w\} \in E(T)$. Then we set $T' = T \setminus \{v\}$ and

$$U'_x = \begin{cases} U_v \otimes_1 U_w, & x = w \\ U_x, & x \neq w \end{cases}$$

and $\mathcal{C}_T((U_x)_{x\in V}) := \mathcal{C}_{T'}((U'_x)_{x\in V\setminus\{v\}})$. Repeat this procedure until $\mathcal{C}_T((U_x)_{x\in V})$ becomes a scalar.

Remark. The definition of $\mathcal{C}_T$ is independent of the order of contraction. First we assume $U_v = u_v^{\otimes d_T(v)}$ with $u_v \in \mathcal{H}$. In this case $\mathcal{C}_T((U_v)_v) = \prod_{e=\{v,w\}\in E(T)} \langle u_v, u_w\rangle$ and this is independent of the order of contraction. By multilinearity, the same holds for finite linear combinations of such symmetric tensors.

Since 1-contraction satisfies $\|g \otimes_1 h\|_{\mathcal{H}^{\otimes(q+r-2)}} \leq \|g\|_{\mathcal{H}^{\otimes q}} \|h\|_{\mathcal{H}^{\otimes r}}$, we have

$$|\mathcal{C}_T((U_v)_{v\in V})| \leq \prod_{v\in V} \|U_v\|_{\mathcal{H}^{\otimes d_T(v)}}. \tag{23}$$

Let $\mathcal{C}_T^{(1)}((U_v)_{v\in V})$ and $\mathcal{C}_T^{(2)}((U_v)_{v\in V})$ be two scalars obtained using two different orders of contraction. Let $U_v \in \mathcal{H}^{\odot d_T(v)}$ be a symmetric tensor. By the polarization identity and density, we can choose a sequence $(U_v^{(n)})_{n\in\mathbb{N}} \subset \mathcal{H}^{\otimes d_T(v)}$ of finite linear combinations of tensors $u^{\otimes d_T(v)}$ with $\|U_v - U_v^{(n)}\|_{\mathcal{H}^{\otimes d_T(v)}} \to 0$ as $n \to \infty$ for $v = 0, i, j, k$. Then

$$\mathcal{C}_T^{(1)}((U_v)_{v\in V}) = \lim_{n\to\infty} \mathcal{C}_T^{(1)}((U_v^{(n)})_{v\in V}) = \lim_{n\to\infty} \mathcal{C}_T^{(2)}((U_v^{(n)})_{v\in V}) = \mathcal{C}_T^{(2)}((U_v)_{v\in V})$$

by (23).

For $\mathbf{c} = (c_0, c_i, c_j, c_k)$, use the evaluation

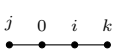
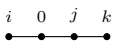
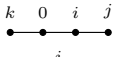
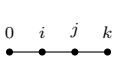
$$[\![T]\!]_{\mathbf{c}} := \mathcal{C}_T(F; c) := \mathcal{C}_T\big(F_0^{[d_T(0)],c_0}, F_i^{[d_T(i)],c_i}, F_j^{[d_T(j)],c_j}, F_k^{[d_T(k)],c_k}\big). \tag{24}$$

The iterated gamma identity (22) is evaluated using the weighted contraction of some trees.

Proposition 22 (Six-term Mehler representation)**.** *Under the assumptions of Lemma 20,*

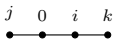
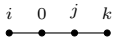
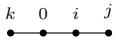
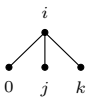
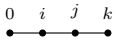
$$\mathbb{E}\{\Gamma(\Gamma(\Gamma(F_0, F_i), F_j), F_k)\} = \int_{(0,\infty)^3} a_1 a_2 a_3 \sum_{\ell=1}^{6} w_\ell\, \mathbb{E}\{[\![T_\ell]\!]_{\mathbf{c}}\}\, d\tau_1 d\tau_2 d\tau_3, \tag{25}$$

where the expectation is over all four Gaussian components and the trees and weights are as follows. An entry uv denotes the edge $\{u, v\}$.

ℓ	T_ℓ	$E(T_\ell)$	w_ℓ
1		$\{0i, 0j, 0k\}$	$a_1^2 a_2$
2		$\{0i, 0j, ik\}$	$a_1 a_2$
3		$\{0i, 0j, jk\}$	a_1
4		$\{0i, ij, 0k\}$	$a_1 a_2$
5		$\{0i, ij, ik\}$	a_2
6		$\{0i, ij, jk\}$	1

All integrals are absolutely convergent.

Remark. Intuitively, we begin with the graph which corresponds to $\Gamma(F_0, F_i)$. The product rule for the inner product gives a_1  and  which correspond to $\langle D(-DL^{-1})F_0, DF_i\rangle$ and $\langle -DL^{-1}F_0, D^2F_i\rangle$ respectively. Operating M_2 and taking inner product with DF_j gives the graphs a_1  and . After doing once more derivative, applying the product rule, operating M_3 and taking the inner product, we have six graphs of the table below.

Parent graph	Free derivative slot	Weight	After M_3 and DF_k
		$a_1(a_1a_2)$	
		a_1a_2	
		$a_1 \cdot 1$	
		$1 \cdot a_1a_2$	
		$1 \cdot a_2$	
		$1 \cdot 1$	

Proof. **Absolute convergence of the integrals.** By (23),

$$\mathbb{E}|\,[\![T_\ell]\!]_{\mathbf{c}}\,| \le \mathbb{E}\prod_{v\in V}\|F_v^{[d_T(v)],c_v}\|_{\mathcal{H}^{\otimes d_T(v)}} \le \prod_{v\in V}(\mathbb{E}\|D^{d_T(v)}F_v\|^4_{\mathcal{H}^{\otimes d_T(v)}})^{1/4}.$$

Since $F_0, \dots, F_3 \in \bigcap_{1<p<\infty}\mathbb{D}^{3,p}$, the right hand side is finite. Note that $0 \le w_l \le 1$ for $l = 1, \dots, 6$ and $\int_{(0,\infty)^3} a_1a_2a_3 d\tau_1 d\tau_2 d\tau_3 = 1$.

Derivation of the formula. First $D^{(0)}F_i = F_i^{[1],e_0}$ and the transfer formula gives

$$D^{(0)}F_0^{[r],R_2R_1e_0} = a_1a_2F_0^{[r+1],R_2R_1e_0}, \qquad D^{(0)}F_i^{[r],R_2e_0} = a_2F_i^{[r+1],R_2e_0}, \qquad D^{(0)}F_j^{[r],e_0} = F_j^{[r+1],e_0}$$

for $r = 1, 2$.

The first gamma operation becomes

$$\begin{aligned}\Gamma_0(F_0, F_i) &= \int_0^\infty d\tau_1 a_1 \mathbb{E}_1\langle M_1(D^{(0)}F_0), D^{(0)}F_i\rangle_{\mathcal{H}},\\ M_1(D^{(0)}F_0) &= M_1(\psi_{D^{(0)}F_0}(W^{(0)})) = \psi_{DF_0}(a_1W^{(0)} + b_1W^{(1)}) = F_0^{[1],R_1e_0},\end{aligned}$$

(these equations are in the sense of μ-a.s. equivalence classes). Hence

$$\Gamma_0(F_0, F_i) = \int_0^\infty d\tau_1 a_1 \mathbb{E}_1\langle F_0^{[1],R_1e_0}, F_i^{[1],e_0}\rangle_{\mathcal{H}}.$$

For the second gamma operation, we need to calculate

$$\begin{aligned}A_0\Gamma(F_0, F_i) &= \int_0^\infty d\tau_2 a_2 \mathbb{E}_2\{M_2D^{(0)}\Gamma_0(F_0, F_i)\},\\ D^{(0)}\Gamma_0(F_0, F_i) &= \int_0^\infty d\tau_1 a_1 \mathbb{E}_1\{\langle D^{(0)}F_0^{[1],R_1e_0}, F_i^{[1],e_0}\rangle_{\mathcal{H}} + \langle F_0^{[1],R_1e_0}, D^{(0)}F_i^{[1],e_0}\rangle_{\mathcal{H}}\},\\ M_2\langle D^{(0)}F_0^{[1],R_1e_0}, F_i^{[1],e_0}\rangle_{\mathcal{H}} &= a_1\langle\psi_{D^2F_0}(W^{R_2R_1e_0}), \psi_{DF_i}(W^{R_2e_0})\rangle_{\mathcal{H}} = a_1\langle F_0^{[2],R_2R_1e_0}, F_i^{[1],R_2e_0}\rangle_{\mathcal{H}},\\ M_2\langle F_0^{[1],R_1e_0}, D^{(0)}F_i^{[1],e_0}\rangle_{\mathcal{H}} &= \langle\psi_{DF_0}(W^{R_2R_1e_0}), \psi_{D^2F_i}(W^{R_2e_0})\rangle_{\mathcal{H}} = \langle F_0^{[1],R_2R_1e_0}, F_i^{[2],R_2e_0}\rangle_{\mathcal{H}},\end{aligned}$$

Hence

$$\begin{aligned}A_0\Gamma_0(F_0, F_i) &= \int_{(0,\infty)^2} d\tau_1 d\tau_2 a_1a_2 \mathbb{E}_1\{\mathbb{E}_2\{a_1F_0^{[2],R_2R_1e_0} \otimes_1 F_i^{[1],R_2e_0} + F_0^{[1],R_2R_1e_0} \otimes_1 F_i^{[2],R_2e_0}\}\},\\ \Gamma(\Gamma_0(F_0, F_i), F_j) &= \int_{(0,\infty)^2} d\tau_1 d\tau_2 a_1a_2 \mathbb{E}_1\{\mathbb{E}_2\{a_1(F_0^{[2],R_2R_1e_0} \otimes_1 F_i^{[1],R_2e_0}) \otimes_1 F_j^{[1],e_0}\\ &\qquad + (F_0^{[1],R_2R_1e_0} \otimes_1 F_i^{[2],R_2e_0}) \otimes_1 F_j^{[1],e_0}\}\}.\end{aligned}$$

Now we write out the third gamma operation:

$$\begin{aligned}&D^{(0)}\Gamma_0(\Gamma_0(F_0, F_i), F_j)\\ &= \int_{(0,\infty)^2} a_1a_2 \mathbb{E}_1\mathbb{E}_2\Big\{\\ &\quad a_1^2a_2\,(F_0^{[3],R_2R_1e_0} \otimes_1 F_i^{[1],R_2e_0}) \otimes_1 F_j^{[1],e_0} + a_1a_2\,F_i^{[2],R_2e_0} \otimes_1 (F_0^{[2],R_2R_1e_0} \otimes_1 F_j^{[1],e_0})\\ &\quad + a_1\,F_j^{[2],e_0} \otimes_1 (F_0^{[2],R_2R_1e_0} \otimes_1 F_i^{[1],R_2e_0}) + a_1a_2\,F_0^{[2],R_2R_1e_0} \otimes_1 (F_i^{[2],R_2e_0} \otimes_1 F_j^{[1],e_0})\\ &\quad + a_2\,(F_i^{[3],R_2e_0} \otimes_1 F_0^{[1],R_2R_1e_0}) \otimes_1 F_j^{[1],e_0} + F_j^{[2],e_0} \otimes_1 (F_i^{[2],R_2e_0} \otimes_1 F_0^{[1],R_2R_1e_0})\\ &\Big\}\, d\tau_1 d\tau_2.\end{aligned}$$

The third inverse-generator formula is

$$A_0\Gamma_0(\Gamma_0(F_0, F_i), F_j) = \int_0^\infty a_3 \mathbb{E}_3\{M_3D^{(0)}\Gamma_0(\Gamma_0(F_0, F_i), F_j)\}\, d\tau_3.$$

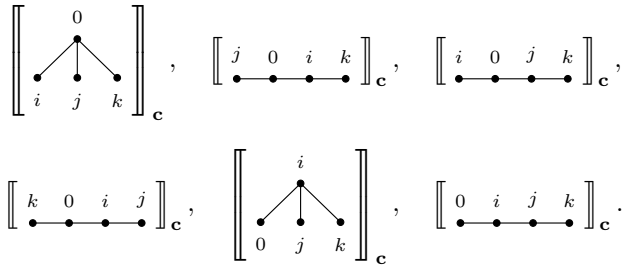
In each of the six terms just displayed, M_3 acts by

$$M_3F_0^{[r],R_2R_1e_0} = F_0^{[r],R_3R_2R_1e_0}, \qquad M_3F_i^{[r],R_2e_0} = F_i^{[r],R_3R_2e_0}, \qquad M_3F_j^{[r],e_0} = F_j^{[r],R_3e_0}.$$

Taking the inner product with $D^{(0)}F_k = F_k^{[1],e_0}$ gives

$$
\begin{aligned}
&\Gamma_0(\Gamma_0(\Gamma_0(F_0,F_i),F_j),F_k)\\
&= \int_{(0,\infty)^3} a_1a_2a_3\mathbb{E}_1\mathbb{E}_2\mathbb{E}_3\Big\{ a_1^2a_2\,\langle F_0^{[3],R_3R_2R_1e_0}, F_i^{[1],R_3R_2e_0}\otimes F_j^{[1],R_3e_0}\otimes F_k^{[1],e_0}\rangle_{\mathcal{H}^{\otimes 3}}\\
&\quad + a_1a_2\,\langle F_i^{[2],R_3R_2e_0}\otimes_1 (F_0^{[2],R_3R_2R_1e_0}\otimes_1 F_j^{[1],R_3e_0}), F_k^{[1],e_0}\rangle_{\mathcal{H}}\\
&\quad + a_1\,\langle F_j^{[2],R_3e_0}\otimes_1 (F_0^{[2],R_3R_2R_1e_0}\otimes_1 F_i^{[1],R_3R_2e_0}), F_k^{[1],e_0}\rangle_{\mathcal{H}}\\
&\quad + a_1a_2\,\langle F_0^{[2],R_3R_2R_1e_0}\otimes_1 (F_i^{[2],R_3R_2e_0}\otimes_1 F_j^{[1],R_3e_0}), F_k^{[1],e_0}\rangle_{\mathcal{H}}\\
&\quad + a_2\,\langle (F_i^{[3],R_3R_2e_0}\otimes_1 F_0^{[1],R_3R_2R_1e_0})\otimes_1 F_j^{[1],R_3e_0}, F_k^{[1],e_0}\rangle_{\mathcal{H}}\\
&\quad + \langle F_j^{[2],R_3e_0}\otimes_1 (F_i^{[2],R_3R_2e_0}\otimes_1 F_0^{[1],R_3R_2R_1e_0}), F_k^{[1],e_0}\rangle_{\mathcal{H}}\Big\}\, d\tau_1 d\tau_2 d\tau_3.
\end{aligned}
$$

In the displayed order, the six scalar contractions are precisely



Their coefficient vectors are $c_0 = R_3R_2R_1e_0$, $c_i = R_3R_2e_0$, $c_j = R_3e_0$, and $c_k = e_0$, as specified before the proposition. The six weights are respectively $a_1^2a_2, a_1a_2, a_1, a_1a_2, a_2, 1$.

Taking $\mathbb{E}_0$ proves

$$
\begin{aligned}
&\mathbb{E}\{\Gamma(\Gamma(\Gamma(F_0,F_i),F_j),F_k)\}\\
&\quad = \int_{(0,\infty)^3} a_1a_2a_3 \sum_{\ell=1}^{6} w_\ell \mathbb{E}\{[\![T_\ell]\!]_{\mathbf{c}}\}\, d\tau_1 d\tau_2 d\tau_3,
\end{aligned}
$$

which is (25).

□

4.2 Application to the stationary solution

An explicit formula for the stationary solution. Fix θ under Assumption 1. For $f_v \in \mathcal{F}_\theta$, Proposition 19 and stationary moments yield $F_v = f_v(\bar{X}_{t_v}^\theta) \in \bigcap_{1<p<\infty}\mathbb{D}^{3,p}$. Suppress θ in Y, J, a and set

$$
U_t^c = \psi_{D\bar{X}_t^\theta}(W^c) = \sigma J_{t,\cdot}^c, \quad V_t^c = \psi_{D^2\bar{X}_t^\theta}(W^c), \quad Z_t^c = \psi_{D^3\bar{X}_t^\theta}(W^c).
$$

Theorem 3 and the transfer formula give

$$
V_t^c = \int_{-\infty}^t a''(Y_r^c) J_{t,r}^c (U_r^c)^{\otimes 2}\, dr, \tag{26}
$$

$$
Z_t^c = \int_{-\infty}^t J_{t,r}^c \left[a'''(Y_r^c)(U_r^c)^{\otimes 3} + 3a''(Y_r^c)\mathrm{Sym}_3(V_r^c\otimes U_r^c)\right] dr. \tag{27}
$$

These Hilbert-tensor identities give

$$
D^{(q)}Y_t^c = c_q U_t^c, \qquad D^{(q_2)}D^{(q_1)}Y_t^c = c_{q_1}c_{q_2}V_t^c.
$$

The chain rule is

$$\begin{aligned}
F_v^{[1],c} &= f_v'(Y_{t_v}^c)U_{t_v}^c,\\
F_v^{[2],c} &= f_v''(Y_{t_v}^c)(U_{t_v}^c)^{\otimes 2} + f_v'(Y_{t_v}^c)V_{t_v}^c,\\
F_v^{[3],c} &= f_v'''(Y_{t_v}^c)(U_{t_v}^c)^{\otimes 3} + 3f_v''(Y_{t_v}^c)\mathrm{Sym}_3(V_{t_v}^c \otimes U_{t_v}^c) + f_v'(Y_{t_v}^c)Z_{t_v}^c.
\end{aligned} \tag{28}$$

Only a', a'', a''' and f_v', f_v'', f_v''' occur. For every finite $p \geq 1$,

$$\sup_{t\in\mathbb{R},\,|c|=1} \mathbb{E}\left[\left(\sum_{r=1}^{3} |f_v^{(r)}(Y_t^c)|\right)^p\right] < \infty. \tag{29}$$

Decorated kernel trees. Let us consider an inner product $\langle V_t^c, U_s^d \otimes U_u^e\rangle_{\mathcal{H}^{\otimes 2}}$ for example. This becomes

$$\langle V_t^c, U_s^d \otimes U_u^e\rangle_{\mathcal{H}^{\otimes 2}} = \int_{-\infty}^{t} a''(Y_r^c) J_{t,r}^c \langle U_r^c, U_s^d\rangle_{\mathcal{H}} \langle U_r^c, U_u^e\rangle_{\mathcal{H}}\, dr$$

and some time variables of the decaying kernel $\langle U_s^c, U_t^d\rangle_{\mathcal{H}}$ are integrated over. In order to incorporate this structure, we consider *decorated kernel trees*. The free vertices are called *external* and the integrated ones *internal*. The solid edges represent $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ (or K below) and dashed ones multiplication by J.

Definition 23 (Kernel diagrams)**.** Set

$$K_{c,d}(s,t) := \langle U_s^c, U_t^d\rangle_{\mathcal{H}} = \Big[\!\Big[\begin{array}{c} s \quad\quad t \\ \bullet\!\!-\!\!-\!\!\bullet \end{array} \Big]\!\Big]_{c,d}^{\ker}. \tag{30}$$

- Fix a tree $T = (V_{\mathrm{ext}}, E(T))$, where $V_{\mathrm{ext}} = \{0, i, j, k\}$. A kernel diagram G associated with T is obtained by adjoining a finite set V_{int} of internal vertices.
- Each internal vertex x has a unique parent $p(x) \in V_{\mathrm{ext}} \cup V_{\mathrm{int}}$, so that the dashed edges $E_d(G) = \{p(x) \to x : x \in V_{\mathrm{int}}\}$ form a forest rooted at the external vertices.
- For $z \in V_{\mathrm{ext}} \sqcup V_{\mathrm{int}}$, let $\lambda(z) \in V_{\mathrm{ext}}$ denote the root of its dashed component. In particular $\lambda(v) = v$ for $v \in V_{\mathrm{ext}}$. Thus every internal vertex inherits the noise attached to its external root.
- The solid edges $E_s(G)$ are in one-to-one correspondence with the edges of the original tree T: for every $e = \{v, w\} \in E(T)$ there is exactly one solid edge $\{x_e, y_e\} \in E_s(G)$ satisfying $\lambda(x_e) = v, \lambda(y_e) = w$. Thus the expansion does not create new contraction edges; it only moves their endpoints along the dashed forests.
- Each internal vertex x carries a label $m_x \in \{2, 3\}$, representing $a^{(m_x)}$, and each external vertex v carries a factor $\phi_v \in \{1, f_v', f_v'', f_v'''\}$.
- Set $D_G(t) = \big\{(r_x)_{x\in V_{\mathrm{int}}} : r_x \leq r_{p(x)} \text{ for every } x \in V_{\mathrm{int}}\big\}$.

Then we define the evaluation of a kernel diagram G by

$$\begin{aligned}
[\![G]\!]_{\mathbf{c}}^{\ker} := \int_{\mathcal{D}_G(\mathbf{t})} &\prod_{v\in V_{\mathrm{ext}}} \phi_v(Y_{t_v}^{c_v}) \prod_{x\in V_{\mathrm{int}}} a^{(m_x)}(Y_{r_x}^{c_{\lambda(x)}})\\
&\times \prod_{x\in V_{\mathrm{int}}} J_{r_{p(x)},r_x}^{c_{\lambda(x)}} \prod_{\{x,y\}\in E_{\mathrm{s}}} K_{c_{\lambda(x)},c_{\lambda(y)}}(r_x, r_y) \prod_{x\in V_{\mathrm{int}}} dr_x.
\end{aligned} \tag{31}$$

Here $r_v = t_v$ for external vertices. Empty products are understood as one.

Remark. Let us explain rules for replacement of vertices and edges. If $d_T(v) = 1$, then

$$F_v^{[1],c_v} = f_v'(Y_{t_v}^{c_v})U_{t_v}^{c_v}.$$

Thus no internal vertex is introduced: the unique solid edge remains attached to v, and the external decoration is $\phi_v = f_v'$.

If $d_T(v) = 2$, then

$$F_v^{[2],c_v} = f_v''(Y_{t_v}^{c_v})(U_{t_v}^{c_v})^{\otimes 2} + f_v'(Y_{t_v}^{c_v})V_{t_v}^{c_v}.$$

The first term leaves both solid edges attached to v, with $\phi_v = f_v''$. For the second term, (26) gives

$$V_{t_v}^{c_v} = \int_{-\infty}^{t_v} a''(Y_r^{c_v})J_{t_v,r}^{c_v}(U_r^{c_v})^{\otimes 2}\,dr.$$

Hence we introduce one internal vertex x, with parent $p(x) = v$ and label $m_x = 2$, and move both solid edges from v to x. In this case $\phi_v = f_v'$. For example,

$$\begin{aligned}\langle V_t^c, U_s^d \otimes U_u^e\rangle_{\mathcal{H}^{\otimes 2}} &= \Bigg[\!\!\Bigg[\begin{array}{c} t \\ \downarrow \\ \circ \\ \diagup\ \diagdown \\ s \qquad u \end{array} \Bigg]\!\!\Bigg]^{\ker}_{c,d,e} \qquad (r;\ a'') \\ &= \int_{-\infty}^{t} a''(Y_r^c)J_{t,r}^c K_{c,d}(r,s)K_{c,e}(r,u)\,dr.\end{aligned} \tag{32}$$

If $d_T(v) = 3$, the three terms in

$$F_v^{[3],c_v} = f_v'''(Y_{t_v}^{c_v})(U_{t_v}^{c_v})^{\otimes 3} + 3f_v''(Y_{t_v}^{c_v})\,\mathrm{Sym}_3\big(V_{t_v}^{c_v} \otimes U_{t_v}^{c_v}\big) + f_v'(Y_{t_v}^{c_v})Z_{t_v}^{c_v}$$

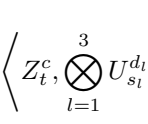
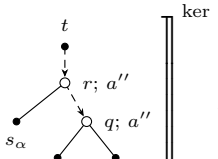
give the following replacements. The f_v'''-term introduces no internal vertex and leaves all three solid edges attached to v. For the f_v''-term, choose one edge $e_0 \in E_T(v)$. The U-factor carrying e_0 remains at v, while the other two U-factors come from $V_{t_v}^{c_v}$. Thus one internal vertex x with $p(x) = v, m_x = 2$ is introduced, and the other two solid edges are moved to x. The three choices of e_0 are precisely the three terms represented by $3\,\mathrm{Sym}_3\big(V_{t_v}^{c_v} \otimes U_{t_v}^{c_v}\big)$. For example,

$$\left\langle 3\,\mathrm{Sym}_3(V_t^c \otimes U_u^d), \bigotimes_{j=1}^{3} U_{s_j}^{e_j} \right\rangle_{\mathcal{H}^{\otimes 3}} = \sum_{(\alpha,\beta,\gamma)\in\mathcal{A}} \Bigg[\!\!\Bigg[\begin{array}{c} u \quad s_\alpha \end{array} \Bigg]\!\!\Bigg]^{\ker}_{d,e_\alpha} \Bigg[\!\!\Bigg[\begin{array}{c} t \\ \downarrow \\ \circ \\ \diagup\ \diagdown \\ s_\beta \qquad s_\gamma \end{array} \Bigg]\!\!\Bigg]^{\ker}_{c,e_\beta,e_\gamma} \qquad (r;\ a''),$$

$$\mathcal{A} = \{(1,2,3),(2,3,1),(3,1,2)\}.$$

Finally, expand $Z_{t_v}^{c_v}$ by (27). Its a'''-term introduces one internal vertex x with $p(x) = v, m_x = 3$, and moves all three incident solid edges to x. For the $a''\,\mathrm{Sym}_3(V \otimes U)$-term, choose again one incident edge e_0. Introduce two internal vertices arranged as $v \longrightarrow x \longrightarrow y, m_x = m_y = 2$. The solid edge labelled by e_0, corresponding to the outer U-factor in $V_r \otimes U_r$, is attached to x, while the other two edges, corresponding to the two U-factors in V_r, are attached to y. Again the three choices of e_0 account

for the symmetrization multiplicity. For example,

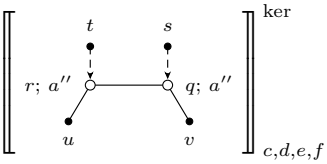
$$\left\langle Z_t^c, \bigotimes_{l=1}^{3} U_{s_l}^{d_l} \right\rangle_{\mathcal{H}^{\otimes 3}} = \Big[\!\Big[$$  $$\Big]\!\Big]_{\boldsymbol{\eta}}^{\ker} + \sum_{(\alpha,\beta,\delta)\in\mathcal{A}} \Big[\!\Big[$$  $$\Big]\!\Big]_{\boldsymbol{\eta}}^{\ker}, \tag{33}$$

$$\mathcal{A} = \{(1,2,3),(2,3,1),(3,1,2)\}.$$

Here $\boldsymbol{\eta} = (c, d_1, d_2, d_3)$ assigns noise c to the root t and noise d_l to the leaf s_l. The nested diagram on the right evaluates to

$$\int_{-\infty}^{t} dr \int_{-\infty}^{r} dq\, a''(Y_r^c) a''(Y_q^c) J_{t,r}^c J_{r,q}^c K_{c,d_\alpha}(r, s_\alpha) K_{c,d_\beta}(q, s_\beta) K_{c,d_\delta}(q, s_\delta).$$

Similarly, for the path with noises (e, c, d, f),

$$\mathcal{C}_{\text{u t s v}}(U_u^e, V_t^c, V_s^d, U_v^f) = \Big[\!\Big[$$  $$\Big]\!\Big]_{c,d,e,f}^{\ker} \tag{34}$$

$$= \int_{-\infty}^{t} dr \int_{-\infty}^{s} dq\, a''(Y_r^c) a''(Y_q^d) J_{t,r}^c J_{s,q}^d K_{e,c}(u, r) K_{c,d}(r, q) K_{d,f}(q, v)$$

and so on.

In this way each original edge of T produces exactly one solid edge of the resulting kernel diagram. In particular, $E_s(G) \longrightarrow E(T), \{x, y\} \longmapsto \{\lambda(x), \lambda(y)\}$ is a bijection. The dashed edges are exactly those introduced by the integral representations of V and Z.

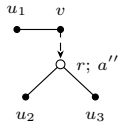
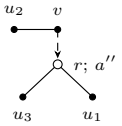
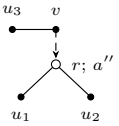
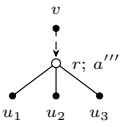
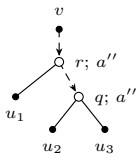
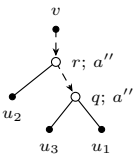
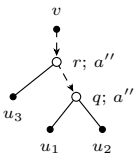
Let $\mathcal{E}(T)$ denote the resulting finite multiset of kernel diagrams. We keep a multiset, rather than a set, because different choices of edge labels in the symmetrized terms may produce the same underlying unlabelled diagram and must retain their multiplicities. With this convention, the preceding expansion gives the following identity.

Lemma 24 (Kernel expansion of a tree contraction)**.** *For every tree T occurring in Proposition 22,*

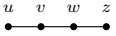
$$[\![T]\!]_{\mathbf{c}} = \sum_{G\in\mathcal{E}(T)} [\![G]\!]_{\mathbf{c}}^{\ker}. \tag{35}$$

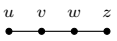
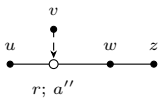
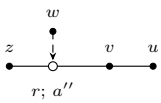
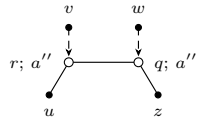
Remark (Complete list of the diagrams in $\mathcal{E}(T)$)*.* Every tree in Proposition 22 is a star or a path. The following lists retain the external labels and the labels of the original edges. Internal vertex names are dummy integration variables; renaming them does not create another element.

Stars. Let $T =$  . Each leaf u_l has decoration $\phi_{u_l} = f'_{u_l}$. The eight elements of $\mathcal{E}(T)$ are drawn below, grouped by the decoration at the centre. The decoration in the left column belongs to every diagram in that row; together with the leaf decorations it completes the data of each decorated kernel tree.

ϕ_v	Elements of $\mathcal{E}(T)$		
f_v'''			
f_v''			
f_v'			
f_v'			

In every diagram, all internal vertices have root v and noise c_v. The counts $1+3+1+3=8$ come respectively from the $f_v'''U^{\otimes 3}$ term, the $3f_v''\mathrm{Sym}_3(V\otimes U)$ term, and the two terms in the expansion of $f_v'Z$.

Paths. Let $T =$ . The end vertices always have $\phi_u = f_u'$ and $\phi_z = f_z'$. At each of v, w, independently, either keep its two solid endpoints there, or move both to a new internal child. The complete list is

Element of $\mathcal{E}(T)$	ϕ_v	ϕ_w
	f_v''	f_w''
	f_v'	f_w''
	f_v''	f_w'
	f_v'	f_w'

The third diagram is drawn in reversed horizontal order to show the expansion at w. Each open vertex has label a'' and inherits the noise of the external vertex above it: in the second diagram r has noise c_v, in the third r has noise c_w, and in the fourth r, q have noises c_v, c_w, respectively. Thus there are $2\cdot 2 = 4$ elements.

Consequently,

$$\#\mathcal{E}(T_1) = \#\mathcal{E}(T_5) = 8, \qquad \#\mathcal{E}(T_l) = 4 \quad (l = 2,3,4,6),$$

where cardinalities count multiplicity and T_l are the six trees in Proposition 22.

4.3 Bounds for Malliavin inner products

Put

$$\gamma = \tfrac{1}{2} - H \in (0, \tfrac{1}{2}), \qquad \beta = 1+\gamma = \tfrac{3}{2} - H, \qquad \rho(u) = (1+|u|)^{-\beta}.$$

The following lemma is a slight generalization of Chiba [2018, Lemma 5.8, pp. 41–43].

Lemma 25 (A bound for one edge)**.** *For fixed unit vectors c, d and $s, t \in \mathbb{R}$,*

$$|\langle U_s^c, U_t^d \rangle_{\mathcal{H}}| \le C\rho(t-s), \tag{36}$$

$$|\langle D^{(q)} Y_s^c, D^{(q)} Y_t^d \rangle_{\mathcal{H}}| \le C|c_q d_q|\rho(t-s), \tag{37}$$

almost surely, where C depends only on $H, \sigma, \alpha, \alpha'$. For varying parameters these bounds hold almost everywhere in their product with probability measure.

Proof. Fix unit vectors c, d and work on the probability-one event on which the paths Y^c, Y^d are continuous. As above, θ is fixed and suppressed. Define

$$q_t^c(x) = J_{t,x}^c = \begin{cases} \exp\left(\int_x^t a'(Y_r^c)\, dr\right), & x \le t, \\ 0, & x > t. \end{cases} \qquad U_t^c = \sigma q_t^c.$$

Thus the dependence on c is through the path Y^c in the exponential. For $x < t$, assumption 1 therefore gives

$$0 \le q_t^c(x) \le e^{-\alpha(t-x)}, \qquad |\partial_x q_t^c(x)| \le \alpha' e^{-\alpha(t-x)}.$$

The constants in these deterministic bounds do not depend on c or on the path. The same statements hold with d in place of c. Set $v = t - x > 0$. A formula for the Marchaud derivative gives

$$|\mathbf{D}_-^\gamma q_t^c(x)| \le C\left[q_t^c(x) v^{-\gamma} + \int_0^v |q_t^c(x) - q_t^c(x+u)| u^{-1-\gamma}\, du\right] \tag{38}$$

(see Nualart [2006, Appendix A.5]). For $0 < v \le 1$, the bracket is bounded by $Cv^{-\gamma} + C\int_0^v u^{-\gamma}\, du \le Cv^{-\gamma}$. For $v \ge 1$, splitting at $v/2$ yields

$$\begin{aligned} \int_0^{v/2} |q_t^c(x) - q_t^c(x+u)| u^{-1-\gamma}\, du &\le Ce^{-\alpha v/2} v^{1-\gamma}, \\ \int_{v/2}^v |q_t^c(x) - q_t^c(x+u)| u^{-1-\gamma}\, du &\le C\int_{v/2}^v (e^{-\alpha v} + e^{-\alpha(v-u)}) u^{-1-\gamma}\, du \\ &\le C(e^{-\alpha v} v^{-\gamma} + v^{-1-\gamma}). \end{aligned}$$

Consequently,

$$|\mathbf{D}_-^\gamma q_t^c(x)| \le Ch(t-x)\mathbf{1}_{\{x<t\}}, \qquad h(v) = \begin{cases} v^{-\gamma}, & 0 < v \le 1, \\ v^{-1-\gamma}, & v > 1, \end{cases} \quad h \in L^1 \cap L^2(0,\infty).$$

Both h and C are independent of c and the path. Apply the bound to q_s^c and q_t^d. By the definition of the inner product on $\mathcal{H}$,

$$\begin{aligned} K_{c,d}(s,t) = \langle U_s^c, U_t^d \rangle_{\mathcal{H}} &= \sigma^2 e_H \int_{\mathbb{R}} (\mathbf{D}_-^\gamma q_s^c)(x)(\mathbf{D}_-^\gamma q_t^d)(x)\, dx, \\ |K_{c,d}(s,t)| &\le C\int_{-\infty}^{s\wedge t} h(s-x)h(t-x)\, dx. \end{aligned}$$

For $\Delta = t - s \geq 0$, substituting $v = s - x$ gives

$$|K_{c,d}(s,t)| \leq C \int_0^\infty h(v)h(v+\Delta)\,dv \leq C \begin{cases} \|h\|_2^2, & 0 \leq \Delta \leq 1, \\ \|h\|_1 \Delta^{-1-\gamma}, & \Delta \geq 1 \end{cases} \leq C\rho(\Delta).$$

The case $t < s$ follows by symmetry. Finally,$D^{(q)}Y_t^c = c_q U_t^c$, which gives (37). □

The next elementary estimate allows us to integrate internal time variables without losing the decay carried by the solid edges.

Lemma 26 (Exponential integration)**.** *For $m \geq 0$, $\beta > 0$, $s_1, \ldots, s_m, t \in \mathbb{R}$,*

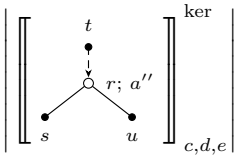
$$\int_{-\infty}^t e^{-\alpha(t-r)} \prod_{l=1}^m (1 + |s_l - r|)^{-\beta}\,dr \leq C_{\alpha,\beta,m} \prod_{l=1}^m (1 + |s_l - t|)^{-\beta}. \tag{39}$$

Proof. For $u = t - r \geq 0$,

$$1 + |s_l - t| \leq (1+u)(1 + |s_l - t + u|), \qquad C_{\alpha,\beta,m} = \int_0^\infty e^{-\alpha u}(1+u)^{m\beta}\,du < \infty.$$

Substitute and multiply over l. □

For example, the diagram (32) can be bounded as



$$\leq C \int_{-\infty}^t e^{-\alpha(t-r)} \rho(r-s)\rho(r-u)\,dr \leq C\rho(t-s)\rho(t-u). \tag{40}$$

(constants C are updated in the second inequality).

Thus integrating the internal vertex simply transfers the decay of the two solid edges from the internal time r to its parent time t. We now formalize this operation.

Definition 27 (Deterministic majorants.)**.** Let $T = (V, E(T))$ be a tree on finite external vertices. Starting from a kernel diagram $G \in \mathcal{E}(T)$, forget the labels m_x, the external decorations ϕ_v, and the noise coefficient vectors. We also allow diagrams obtained from such a graph by repeatedly removing dashed leaves as described below.

Denote this enlarged family by $\overline{\mathcal{E}}(T)$. Every $G \in \overline{\mathcal{E}}(T)$ has the following properties:

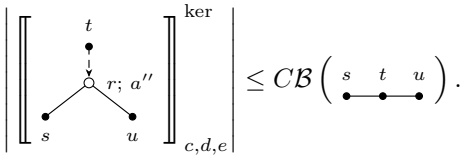
- the dashed edges form a forest rooted at V;
- every vertex z has a root $\lambda(z) \in V$;
- the map $E_s(G) \longrightarrow E(T), \{x,y\} \longmapsto \{\lambda(x), \lambda(y)\}$ is a bijection.

For $G \in \overline{\mathcal{E}}(T)$, set $r_v = t_v$ for $v \in V$ and define

$$\mathcal{B}(G) := \int_{D_G(\mathbf{t})} \prod_{p(x) \to x \in E_d(G)} e^{-\alpha(r_{p(x)} - r_x)} \prod_{\{x,y\} \in E_s(G)} \rho(r_x - r_y) \prod_{x \in V_{\mathrm{int}}(G)} dr_x.$$

This is the deterministic majorant obtained from the kernel evaluation by replacing every propagation factor J with its exponential bound and every solid kernel K with ρ. The estimate (40) is rewritten

using $\mathcal{B}$ as



Lemma 28 (Pruning a dashed leaf)**.** *Let $G \in \overline{\mathcal{E}}(T)$, and let $x \in V_{\mathrm{int}}(G)$ have no dashed children. Put $m = \#\{e \in E_s(G) : x \in e\}$. Remove x and the dashed edge $p(x) \to x$, and move every solid edge attached to x to its parent $p(x)$. Denote the resulting graph by G/x. Then $G/x \in \overline{\mathcal{E}}(T)$ and $\mathcal{B}(G) \le C_{\alpha,\beta,m}\mathcal{B}(G/x)$.*

Proof. Since x and $p(x)$ belong to the same dashed component, $\lambda(x) = \lambda(p(x))$. Hence moving a solid edge from x to $p(x)$ does not change the corresponding edge of the original tree T. In particular, the bijection $E_s(G) \longrightarrow E(T), \{y, z\} \longmapsto \{\lambda(y), \lambda(z)\}$, is preserved, and therefore $G/x \in \overline{\mathcal{E}}(T)$.

Let $y_1, \dots, y_m$ be the solid neighbors of x. Since x has no dashed children, after fixing all time variables other than r_x, the factors in $\mathcal{B}(G)$ which depend on r_x are precisely

$$e^{-\alpha(r_{p(x)} - r_x)} \prod_{j=1}^{m} \rho(r_x - r_{y_j}), \qquad r_x \le r_{p(x)}.$$

Lemma 26 therefore gives

$$\int_{-\infty}^{r_{p(x)}} e^{-\alpha(r_{p(x)} - r_x)} \prod_{j=1}^{m} \rho(r_x - r_{y_j})\, dr_x \le C_{\alpha,\beta,m} \prod_{j=1}^{m} \rho(r_{p(x)} - r_{y_j}).$$

The product on the right-hand side is exactly the contribution obtained by moving all solid edges attached to x to $p(x)$. The remaining factors are independent of r_x, and hence (after changing the order of integration) $\mathcal{B}(G) \le C_{\alpha,\beta,m}\mathcal{B}(G/x)$. □

Every nonempty finite dashed forest has a vertex with no dashed children. Thus Lemma 28 can be applied repeatedly until all internal vertices are removed.

Corollary 29 (Removal of all internal vertices)**.** *For every $G \in \mathcal{E}(T)$,*

$$\mathcal{B}(G) \le C \prod_{\{v,w\} \in E(T)} \rho(t_v - t_w),$$

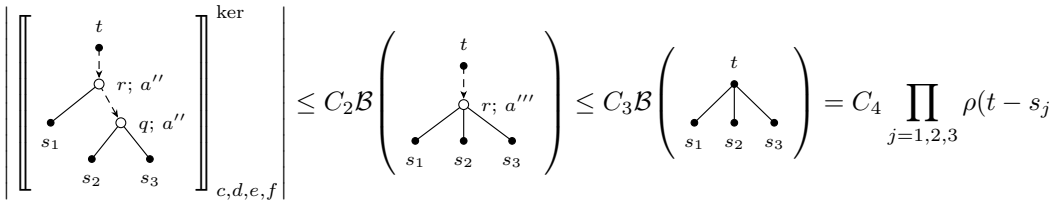
where C is independent of the external times and the coefficient vectors.

Proof. Starting from G, repeatedly choose a dashed leaf and apply Lemma 28. Since the dashed forest is finite, after finitely many steps all internal vertices are removed. The root labels of the endpoints of every solid edge are preserved at each step, so the final graph is the original tree T. Consequently,

$$\mathcal{B}(G) \le C\, \mathcal{B}(T) = C \prod_{\{v,w\} \in E(T)} \rho(t_v - t_w).$$

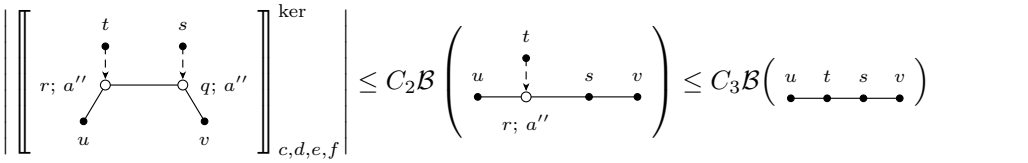
Moreover, the local replacement rules of Section 4.2 show that every $G \in \mathcal{E}(T)$ has at most two internal vertices. Indeed, a tree on four vertices has degree sequence either $(3, 1, 1, 1)$ or $(2, 2, 1, 1)$. A vertex of degree one introduces no internal vertex, a vertex of degree two introduces at most one, and a vertex of degree three introduces at most two. Hence the number of applications of Lemma 28 is uniformly bounded. □

Remark. For example, the diagrams in (33) satisfy



$$\leq C_2\mathcal{B}\left(\quad\right) \leq C_3\mathcal{B}\left(\quad\right) = C_4 \prod_{j=1,2,3} \rho(t - s_j).$$

Another example is the diagram (34):



$$\leq C_2\mathcal{B}\left(\quad\right) \leq C_3\mathcal{B}\left(\quad\right)$$

$$\leq C_4\rho(u - t)\rho(t - s)\rho(s - v).$$

We now return to the random kernel evaluations.

Lemma 30 (Contractions attached to a tree)**.** *Let T be a tree on four vertices, let $F_v = f_v(\bar{X}^{\theta}_{t_v})$, and let c_v be arbitrary unit coefficient vectors on the same finite product Gaussian space. Then*

$$E\left[|[\![T]\!]_{\mathbf{c}}|\right] \leq C \prod_{\{v,w\}\in E(T)} \rho(t_v - t_w),$$

where C is independent of the times and the coefficient vectors.

Proof. By Lemma 24, $[\![T]\!]_{\mathbf{c}} = \sum_{G\in\mathcal{E}(T)} [\![G]\!]^{\mathrm{ker}}_{\mathbf{c}}$. Fix $G \in \mathcal{E}(T)$, and denote by $I_G(r)$ the integrand defining $[\![G]\!]^{\mathrm{ker}}_{\mathbf{c}}$. For every external vertex v, its decoration is of the form $\phi_v = f_v^{(\ell_v)}$, $1 \leq \ell_v \leq 3$. Set

$$A_G := \prod_{v\in V} \left| f_v^{(\ell_v)}(Y^{c_v}_{t_v}) \right|.$$

By boundedness of a'' and a''', Lemma 25 and $0 \leq J^{c_{\lambda(x)}}_{r_{p(x)},r_x} \leq e^{-\alpha(r_{p(x)} - r_x)}$, we have almost surely

$$\int_{D_G(t)} |I_G(r)| \prod_{x\in V_{\mathrm{int}}} dr_x \leq C A_G \mathcal{B}(G) \leq C A_G \prod_{\{v,w\}\in E(T)} \rho(t_v - t_w).$$

Hölder's inequality and (26) give

$$E[A_G] \leq \prod_{v\in V} \left\| f_v^{(\ell_v)}(Y^{c_v}_{t_v}) \right\|_{L^4} \leq C,$$

uniformly in the external times and in the unit coefficient vectors. Hence

$$E\left[\left|[\![G]\!]^{\mathrm{ker}}_{\mathbf{c}}\right|\right] \leq C \prod_{\{v,w\}\in E(T)} \rho(t_v - t_w).$$

Since $\mathcal{E}(T)$ is a finite multiset whose multiplicities depend only on the local expansions of Section 4.2, summing over $G \in \mathcal{E}(T)$ proves the assertion. □

It remains to pass from the decay associated with the edges of an arbitrary tree to the decay associated with the three consecutive time gaps.

Lemma 31 (From tree edges to consecutive gaps)**.** *Let $t_0, \dots, t_3$ be distinct, and let $t_{(0)} < t_{(1)} < t_{(2)} < t_{(3)}$ be their increasing rearrangement. For every tree T on $\{0,1,2,3\}$,*

$$\prod_{\{v,w\}\in E(T)} \rho(t_v - t_w) \le \prod_{\ell=0}^{2} \rho\big(t_{(\ell+1)} - t_{(\ell)}\big).$$

Proof. Let v_0, v_1, v_2, v_3 be the vertices ordered so that

$$t_{v_0} = t_{(0)} < t_{v_1} = t_{(1)} < t_{v_2} = t_{(2)} < t_{v_3} = t_{(3)},$$

and let P be the path consisting of the three adjacent edges $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$. If $T = P$, there is nothing to prove. Otherwise, choose an edge

$$e = \{v_\ell, v_{\ell+1}\} \in E(P) \setminus E(T).$$

Adding e to T creates a unique cycle. This cycle contains an edge $f = \{v_r, v_s\} \in E(T)$ crossing the gap $(t_{(\ell)}, t_{(\ell+1)})$. Hence $|t_{v_r} - t_{v_s}| \ge t_{(\ell+1)} - t_{(\ell)}$. Since ρ is decreasing on $[0,\infty)$,

$$\rho(t_{v_r} - t_{v_s}) \le \rho\big(t_{(\ell+1)} - t_{(\ell)}\big).$$

Replacing f by e therefore produces another tree T' such that

$$\prod_{\{v,w\}\in E(T)} \rho(t_v - t_w) \le \prod_{\{v,w\}\in E(T')} \rho(t_v - t_w).$$

Moreover, T' contains one more edge of P than T. Repeating this operation finitely many times yields P, and hence

$$\prod_{\{v,w\}\in E(T)} \rho(t_v - t_w) \le \prod_{\ell=0}^{2} \rho\big(t_{(\ell+1)} - t_{(\ell)}\big)$$

holds. □

Proof of Theorem 4. Set $F_v = f_v(\bar{X}^\theta_{t_v})$. By stationarity and $f_v \in \mathcal{F}_\theta$, $E[F_v] = 0$. Lemma 20 and Proposition 22 give

$$\kappa_4(F_0, F_1, F_2, F_3) = \sum_{(i,j,k)\in S_3} \int_{(0,\infty)^3} a_1 a_2 a_3 \sum_{\ell=1}^{6} w_\ell E\left[[\![T_\ell]\!]_{\mathbf{c}}\right] d\tau_1 d\tau_2 d\tau_3.$$

By Lemma 30,

$$E\left[|[\![T_\ell]\!]_{\mathbf{c}}|\right] \le C \prod_{\{v,w\}\in E(T_\ell)} \rho(t_v - t_w).$$

Lemma 31 then yields

$$E\left[|[\![T_\ell]\!]_{\mathbf{c}}|\right] \le C \prod_{r=0}^{2} \rho\big(t_{(r+1)} - t_{(r)}\big).$$

Since $0 \le w_\ell \le 1$ and $\int_{(0,\infty)^3} a_1 a_2 a_3 \, d\tau_1 d\tau_2 d\tau_3 = 1$, we obtain

$$|\kappa_4(F_0, F_1, F_2, F_3)| \le C \prod_{r=0}^{2} \rho\big(t_{(r+1)} - t_{(r)}\big).$$

Recalling that $\rho(u) = (1 + |u|)^{H-3/2}$, the desired estimate follows. □

5 The LAN Property

Fix $\theta \in \Theta$ and $H \in (0, 1/2)$. Throughout this section, Assumptions 1 and 2 are in force and

$$f_i(x) = (\partial_{\theta_i} a)(x, \theta), \qquad \mathbb{E}\{f_i(\bar{X}_0^\theta)\} = 0, \qquad i = 1, \dots, m.$$

Thus $f_i \in \mathcal{F}_\theta$. Constants may depend on the fixed θ, H, σ and the functions f_i, but not on T. The initial value x is deterministic and independent of the parameter. We follow the likelihood reduction of Chiba [2018, Sections 4–5] and the variance decomposition of Chiba and Takabatake [2024, Proposition 6.1 and Section 6.6]. The latter uses Gaussianity of the fOU process. Here the additional fourth-cumulant contribution is controlled using Theorem 4.

Likelihood expansion. On $C([0,T])$, write π_t for the coordinate process and set

$$B_t^\theta = \sigma^{-1}\left(\pi_t - x - \int_0^t a(\pi_s, \theta)\, ds\right).$$

Under μ_θ^T, this is a standard fBM. Let W^θ be its innovation Brownian motion in the completed natural filtration on $[0,T]$, obtained by the inverse Volterra transform. This Brownian motion is distinct from the isonormal notation used in Sections 3–4. Put $c_H = \sigma^{-1}\bar{d}_H^{-1}$ and define, for a fixed observed path,

$$\beta_t(\vartheta) = c_H t^{H-1/2} \int_0^t (t-s)^{-H-1/2} s^{1/2-H} a(\pi_s, \vartheta)\, ds, \tag{41}$$

$$q_t(\theta) = (\partial_\theta \beta_t)(\theta) = c_H \int_0^t (t-s)^{-H-1/2} (s/t)^{1/2-H} (\partial_\theta a)(\pi_s, \theta)\, ds. \tag{42}$$

Here $q_t(\theta)$ is a column vector; the derivative is taken with π fixed, not through the solution $X^{\theta,x}$.

The likelihood formula to use is

$$\log \frac{d\mu_\vartheta^T}{d\mu_\theta^T} = \int_0^T \{\beta_t(\vartheta) - \beta_t(\theta)\}\, dW_t^\theta - \frac{1}{2}\int_0^T \{\beta_t(\vartheta) - \beta_t(\theta)\}^2\, dt. \tag{43}$$

See Chiba [2018, Theorem 2.8(1), Definition 2.6]; its likelihood statement covers $0 < H < 1/2$.

For fixed $u \in \mathbb{R}^m$, let $\theta_T = \theta + T^{-1/2}u$; then $\theta_T \in U$ for all sufficiently large T. Define

$$\Delta_\theta^T = T^{-1/2}\int_0^T q_t(\theta)\, dW_t^\theta, \qquad I_T(\theta) = T^{-1}\int_0^T q_t(\theta) q_t(\theta)^\star\, dt, \tag{44}$$

$$R_t^T(u) = \beta_t(\theta_T) - \beta_t(\theta) - T^{-1/2} u^\star q_t(\theta). \tag{45}$$

Proposition 32 (Conditional LAN reduction)**.** *Suppose that* (43) *holds, that $q(\theta)$ is progressively measurable and square integrable on each finite interval, and that*

$$I_T(\theta) \longrightarrow I(\theta) \quad \textit{in probability}, \qquad \mathbb{E}^{\mu_\theta^T}\int_0^T |R_t^T(u)|^2\, dt \longrightarrow 0 \tag{46}$$

for each fixed u. Then the LAN expansion of Theorem 6 holds with $\varphi_T(\theta) = T^{-1/2}E_m$, in the positive-semidefinite convention of Section 2.

Proof. Substitution in (43) gives

$$\log Z_\theta^T(u) = u^\star \Delta_\theta^T - \tfrac{1}{2} u^\star I(\theta) u + r_{\theta,u}^T, \tag{47}$$

$$r_{\theta,u}^T = -\tfrac{1}{2} u^\star \{I_T(\theta) - I(\theta)\} u + \int_0^T R_t^T(u)\, dW_t^\theta - T^{-1/2} \int_0^T u^\star q_t(\theta) R_t^T(u)\, dt - \tfrac{1}{2} \int_0^T |R_t^T(u)|^2\, dt. \tag{48}$$

The bracket of the score martingale is $I_T(\theta)$. The continuous-martingale central limit theorem (see Chiba, 2018, Theorem 5.1) and Cramér–Wold give $\Delta_\theta^T \Rightarrow \mathcal{N}_m(0, I(\theta))$. For a direction with limiting variance zero, the scalar martingale converges to zero in probability by its bracket. Itô's isometry and

$$\left| T^{-1/2} \int_0^T u^\star q_t(\theta) R_t^T(u)\, dt \right| \le \{u^\star I_T(\theta) u\}^{1/2} \left(\int_0^T |R_t^T(u)|^2\, dt \right)^{1/2}$$

show that $r_{\theta,u}^T \to 0$ in probability. □

Lemma 33 (Taylor residual)**.** *For each fixed u and all sufficiently large T,*

$$\mathbb{E}^{\mu_\theta^T} |R_t^T(u)|^2 \le C_u T^{-2} t^{1-2H}, \qquad \mathbb{E}^{\mu_\theta^T} \int_0^T |R_t^T(u)|^2\, dt \le C_u T^{-2H} \longrightarrow 0. \tag{49}$$

Proof. By (A2), the Taylor residual of $a(z, \theta + T^{-1/2}u)$ is bounded by $C_u T^{-1}(1 + |z|^p)$. The fixed-parameter moment bounds of $X^{\theta,x}$ and Minkowski's inequality imply

$$\|R_t^T(u)\|_{L^2(\mu_\theta^T)} \le C_u T^{-1} t^{H-1/2} \int_0^t (t-s)^{-H-1/2} s^{1/2-H}\, ds = C_u T^{-1} B(1/2 - H, 3/2 - H) t^{1/2-H}.$$

Square and integrate. Only moments at the fixed true parameter are used. □

Law of large numbers for the quadratic term. In the following calculations, $X^{\theta,x}$ and $\bar{X}^\theta$ are coupled using the same two-sided fBM; expectations are under $\mathbb{P}$ on this space. Pulling back coordinate functionals through $X^{\theta,x}$ gives their law under μ_θ^T. Write $F_i(t) = f_i(\bar{X}_t^\theta)$ and set

$$Y_i(t) = \int_0^t r^{-H-1/2} F_i(t-r)\, dr, \tag{50}$$

$$E_i(t) = \int_0^t r^{-H-1/2} \{(1 - r/t)^{1/2-H} - 1\} F_i(t-r)\, dr. \tag{51}$$

The stationary version $\bar{q}_i(t)$ of (42) satisfies $\bar{q}_i(t) = c_H\{Y_i(t) + E_i(t)\}$.

Lemma 34 (Second-order reduction)**.** *For $\widetilde{I}_{T,ij} = c_H^2 T^{-1} \int_0^T Y_i(t) Y_j(t)\, dt$,*

$$I_{T,ij}(\theta)(X^{\theta,x}) - \widetilde{I}_{T,ij} \longrightarrow 0 \quad \text{in } L^1(\mathbb{P}), \qquad \mathbb{E}\widetilde{I}_{T,ij} \longrightarrow I_{ij}(\theta). \tag{52}$$

Proof. First note that

$$|\mathbb{E}\{F_i(s) F_j(t)\}| \le C\rho(t-s), \qquad \rho(v) = (1 + |v|)^{H-3/2} \tag{53}$$

holds as a corollary of Lemma 25.

The integrable majorant needed below is

$$\int_0^\infty \int_0^\infty r^{-H-1/2} v^{-H-1/2} \rho(r-v)\, dv\, dr = 2B(1/2-H, 2H) \int_0^\infty z^{-2H} \rho(z)\, dz < \infty. \tag{54}$$

Indeed, on $r > v$, put $r = v + z$ and then $v = zw$. Consequently $\sup_t \mathbb{E}|Y_i(t)|^2 < \infty$. The multiplier in braces in (51) is bounded by one in absolute value and tends to zero for fixed r as $t \to \infty$. Dominated convergence on $(0,\infty)^2$ and Cesàro averaging give

$$\mathbb{E}|E_i(t)|^2 \longrightarrow 0, \qquad T^{-1} \int_0^T \mathbb{E}|E_i(t)|^2\, dt \longrightarrow 0.$$

By (A1), (A3), and the coupling bound $|X_s^{\theta,x} - \bar{X}_s^\theta| \le e^{-\alpha s}|x - \bar{X}_0^\theta|$,

$$\|q_i(t)(X^{\theta,x}) - \bar{q}_i(t)\|_2 \le C \int_0^t (t-s)^{-H-1/2} e^{-\alpha s}\, ds \le C \begin{cases} t^{1/2-H}, & 0 < t \le 1, \\ t^{-H-1/2}, & t > 1. \end{cases}$$

Thus the squared difference is integrable in $t \in (0,\infty)$. Cauchy–Schwarz proves the first assertion. For the second, stationarity, dominated convergence using (54), and Cesàro averaging yield

$$\lim_{T\to\infty} \mathbb{E}\widetilde{I}_{T,ij} = c_H^2 \int_0^\infty \int_0^\infty r^{-H-1/2} v^{-H-1/2} \mathbb{E}\{F_i(-r)F_j(-v)\}\, dv\, dr = I_{ij}(\theta).$$

For the last equality, shift both times by $r + v$ and interchange r, v. □

For fixed i, j, retain the notation

$$\mathtt{I}_T = T^{-1} \int_0^T \{Y_i(t)Y_j(t) - \mathbb{E}[Y_i(t)Y_j(t)]\}\, dt. \tag{55}$$

Let $\mathcal{D}_T$ consist of $0 < t_1, t_2 < T$, $0 < r_1, u_1 < t_1$, $0 < r_2, u_2 < t_2$, and write

$$d\lambda = dt_1\, dt_2\, dr_1\, du_1\, dr_2\, du_2, \qquad K = \prod_{\ell=1}^{2} (t_\ell - r_\ell)^{-H-1/2} (t_\ell - u_\ell)^{-H-1/2}.$$

Define

$$\begin{aligned} C_1 &= \mathbb{E}[F_i(r_1)F_i(r_2)]\, \mathbb{E}[F_j(u_1)F_j(u_2)], \\ C_2 &= \mathbb{E}[F_i(r_1)F_j(u_2)]\, \mathbb{E}[F_j(u_1)F_i(r_2)], \\ C_3 &= \kappa_4(F_i(r_1), F_j(u_1), F_i(r_2), F_j(u_2)). \end{aligned}$$

The centred fourth-moment identity gives

$$\mathbb{E}|\mathtt{I}_T|^2 = \sum_{\ell=1}^{3} \mathtt{I}_T^{(\ell)}, \qquad \mathtt{I}_T^{(\ell)} = T^{-2} \int_{\mathcal{D}_T} K C_\ell\, d\lambda. \tag{56}$$

The individual terms need not be nonnegative. At fixed T, Fubini is justified by uniform fourth moments and local integrability of $r^{-H-1/2}$. A proof of LAN is completed if we can verify $\mathtt{I}_T^{(\ell)} = o(1)$.

The terms with products of covariances. By (53), both covariance-product terms satisfy

$$|\mathtt{I}_T^{(1)}| + |\mathtt{I}_T^{(2)}| \le \frac{C}{T^2} \int_0^T \int_0^T A(t_1,t_2)^2 \, dt_2 \, dt_1, \tag{57}$$

$$A(t_1,t_2) = \int_0^{t_1} \int_0^{t_2} (t_1 - s)^{-H-1/2} (t_2 - v)^{-H-1/2} \rho(s-v) \, dv \, ds. \tag{58}$$

The second product has the same majorant after interchanging r_2, u_2.

Lemma 35. *We have*

$$\frac{1}{T^2} \int_0^T \int_0^T A(t_1,t_2)^2 \, dt_2 dt_1 = o(1). \tag{59}$$

Proof. Indeed, by symmetry it suffices to consider $t_1 \ge t_2$. Put

$$d = t_1 - t_2, \qquad x = t_1 - s, \qquad y = t_2 - v, \qquad a = H + \frac{1}{2}.$$

Since the integrand is nonnegative,

$$A(t_1,t_2) \le \int_0^\infty \int_0^\infty x^{-a} y^{-a} \rho(d - x + y) \, dy \, dx.$$

Changing variables $z = x - y$ and integrating along the lines $x - y = z$, we obtain

$$A(t_1,t_2) \le \int_{\mathbb{R}} \rho(d-z) \int_{\max(0,-z)}^\infty (y+z)^{-a} y^{-a} dy \, dz \le C_H \int_{\mathbb{R}} |z|^{-2H} \rho(d-z) \, dz,$$

because

$$\int_0^\infty (y + |z|)^{-a} y^{-a} \, dy = B(1-a, 2a-1) |z|^{1-2a} = C_H |z|^{-2H}.$$

Recall that $\rho(z) = (1+|z|)^{-\beta}$ with $\beta = 3/2 - H$. Since $0 < 2H < 1$ and $\beta > 1$, the last convolution is uniformly bounded for $0 \le d \le 1$. For $d \ge 1$, split $\mathbb{R}$ into

$$\{|z| \le d/2\}, \qquad \{|d-z| \le d/2\}, \qquad \{|z| > d/2, \ |d-z| > d/2\}.$$

On these three regions, respectively,

$$\int_{|z| \le d/2} |z|^{-2H} \rho(d-z) \, dz \le C d^{-\beta} \int_{|z| \le d/2} |z|^{-2H} \, dz \le C d^{-H-1/2},$$
$$\int_{|d-z| \le d/2} |z|^{-2H} \rho(d-z) \, dz \le C d^{-2H} \int_{\mathbb{R}} \rho(w) \, dw \le C d^{-2H},$$

while, on the last region, necessarily $z < -d/2$ or $z > 3d/2$, and hence

$$\int_{\substack{|z|>d/2 \\ |d-z|>d/2}} |z|^{-2H} \rho(d-z) \, dz \le C \int_{d/2}^\infty r^{-2H-\beta} \, dr \le C d^{-H-1/2}.$$

Since $H < 1/2$, we have $d^{-H-1/2} \le d^{-2H}$ for $d \ge 1$. Consequently,

$$A(t_1,t_2) \le C_H (1 + |t_1 - t_2|)^{-2H}.$$

It follows that

$$\frac{1}{T^2}\int_0^T\int_0^T A(t_1,t_2)^2\,dt_2dt_1 \le \frac{C_H}{T^2}\int_0^T\int_0^T (1+|t_1-t_2|)^{-4H}\,dt_2dt_1$$
$$\le \frac{C_H}{T}\int_0^T (1+d)^{-4H}\,dd = o(1),$$

which proves (59). □

Remark. Chiba and Takabatake [2024, Section 6.6] split the region $t_1 \ge t_2$ into $t_1 < 1$, $t_1 - t_2 < 1$, and $t_1 - t_2 \ge 1$; in the last region, split the s integral at t_2. It can be shown that the contributions are $o(1)$ except in the $s > t_2$ region. For the $s > t_2$ region, to show an $o(1)$ contribution, $\rho(t)$ was required to decay as $(1+|t|)^{2H-2}$, which does not hold in the current setting. However this is just caused by a rough estimate of an upper bound. For completeness, we give a sharp estimate of this region in Lemma 36 below.

Lemma 36 (The cross-region estimate)**.** *For $T \ge 1$,*

$$\frac{1}{T^2}\int_1^T\int_0^{t_1-1}\left(\int_{t_2}^{t_1}\int_0^{t_2}(t_1-s)^{-H-1/2}(t_2-v)^{-H-1/2}\rho(s-v)\,dv\,ds\right)^2 dt_2\,dt_1 \le \frac{C(H)}{T}. \tag{60}$$

Proof. Put $d = t_1 - t_2 \ge 1$, $x = t_1 - s$, and $y = t_2 - v$. The inner integral is bounded by

$$\int_0^d x^{-H-1/2}\int_0^\infty y^{-H-1/2}(1+d-x+y)^{H-3/2}\,dy\,dx = B(1/2-H,1)\int_0^d \frac{x^{-H-1/2}}{1+d-x}\,dx$$
$$\le C(H)d^{-H-1/2}\log(1+d).$$

The last bound follows by splitting at $x = d/2$. Hence the left-hand side of (60) is at most

$$\frac{C(H)}{T^2}\int_1^T (T-d)d^{-1-2H}\log^2(1+d)\,dd \le \frac{C(H)}{T}\int_1^\infty d^{-1-2H}\log^2(1+d)\,dd \le \frac{C'(H)}{T}.$$

□

The terms with a joint fourth cumulant.

Lemma 37 (Integration of two fractional kernels)**.** *For $0 \le r < u \le T$,*

$$\int_u^T (t-r)^{-H-1/2}(t-u)^{-H-1/2}\,dt \le B(1/2-H,2H)(u-r)^{-2H}. \tag{61}$$

Proof. Extend the integral to infinity and put $t - u = (u-r)v$. The remaining integral is $\int_0^\infty (1+v)^{-H-1/2}v^{-H-1/2}\,dv = B(1/2-H,2H)$. Both parameters are positive for $0 < H < 1/2$. □

Lemma 38. *We have $|\mathtt{I}_T^{(3)}| = O(T^{-1})$.*

Proof. Relabel (r_1,u_1,r_2,u_2) as (r_1,r_2,r_3,r_4). Theorem 4, Tonelli's theorem for the nonnegative majorant, and Lemma 37 yield

$$|\mathtt{I}_T^{(3)}| \le \frac{C}{T^2}\int_{[0,T]^4}\prod_{k=1}^3 \rho(r_{(k+1)}-r_{(k)})|r_2-r_1|^{-2H}|r_4-r_3|^{-2H}\,d\mathbf{r}. \tag{62}$$

Equal-time hyperplanes have Lebesgue measure zero and may be omitted. On each of the 24 order regions, put $a = r_{(2)} - r_{(1)}$, $b = r_{(3)} - r_{(2)}$, $c = r_{(4)} - r_{(3)}$. Integration over the smallest time contributes

at most T. The three possible pair configurations have singular factors

configuration	typical example	singular factor
separated	$r_1 < r_2 < r_3 < r_4$	$a^{-2H}c^{-2H}$
interlaced	$r_1 < r_3 < r_2 < r_4$	$(a+b)^{-2H}(b+c)^{-2H}$
nested	$r_1 < r_3 < r_4 < r_2$	$(a+b+c)^{-2H}b^{-2H}$.

Each configuration occurs eight times in this symmetric majorant.

To conclude $|\mathtt{I}_T^{(3)}| = O(T^{-1})$, we verify finiteness of

$$\int_{(0,\infty)^3} \rho(a)\rho(b)\rho(c) \begin{cases} a^{-2H}c^{-2H}, \\ (a+b)^{-2H}(b+c)^{-2H}, \\ (a+b+c)^{-2H}b^{-2H} \end{cases} da\, db\, dc \tag{63}$$

for each of the three displayed integrands. All the cases have the same upper bound

$$\left(\int_0^\infty \rho(a)a^{-2H}\, da\right)^2 \int_0^\infty \rho(b)\, db < \infty.$$

□

Disclosure of AI Use

The author used GPT-5.6 Sol, GPT-6 Astra and Claude Fable 5 to discuss this and related problems, and mainly GPT-5.6 Sol to help organize the manuscript. OpenAI Codex also assisted with manuscript writing.

Contributions. The following describes the roles of the AI systems used in developing this manuscript. **In short, all statements and proofs in this paper were developed with heavy reliance on AI.**

The author had been continuously interested in this problem for a long time and had discussed it with AIs many times. One day GPT suggested the papers Li et al. [2023], Li and Sieber [2022], which treat the history part of an fBM as a fixed path by conditioning. The author came up with the idea of treating the independent path ω' as a fixed forcing when differentiating $\omega \mapsto f(\bar{X}_t(e^{-\tau}\omega + \sqrt{1-e^{-2\tau}}\omega'))$, and iterating it (however, this idea is nothing new after all, and is found in Section 1.5 of Nualart [2006] for example). After that the author verified that

- $D^2\bar{X}_t$ becomes an integral of a tensor product of some first order terms, which corresponds to Theorem 3 (2),
- some of the terms can be bounded from above by the product of ρ, which corresponds to (some terms of) Lemma 30, and
- the LAN property holds using the cumulant estimate in Theorem 4,

and wrote a very rough draft of this paper. Based on the draft, GPT and Codex polished the statements and gave proofs, and the author edited them and decided what needed to be done to complete the manuscript. The cycle went on until this paper was completed.

References

C. Amorino, E. Nualart, F. Panloup, and J. Sieber. Fast convergence rates for estimating the stationary density in SDEs driven by a fractional Brownian motion with semi-contractive drift. *The Annals of Statistics*, 2026. URL `https://arxiv.org/abs/2408.15904`. Forthcoming; preprint arXiv:2408.15904.

K. Chiba. LAN property for stochastic differential equations driven by fractional Brownian motion of Hurst parameter $H \in (1/4, 1/2)$. arXiv preprint arXiv:1804.04108v2, 2018. URL `https://arxiv.org/abs/1804.04108v2`.

K. Chiba. An M-estimator for stochastic differential equations driven by fractional Brownian motion with small Hurst parameter. *Statistical Inference for Stochastic Processes*, 23(2):319–353, 2020. doi: 10.1007/s11203-020-09214-4.

K. Chiba and T. Takabatake. Asymptotically efficient estimation of ergodic rough fractional Ornstein–Uhlenbeck process under continuous observations. *Statistical Inference for Stochastic Processes*, 27 (1):103–122, 2024. doi: 10.1007/s11203-023-09300-3. URL `https://arxiv.org/abs/2204.03271v2`. Proposition and section numbers cited here refer to arXiv:2204.03271v2.

R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, Heidelberg, fifth edition, 2017. doi: 10.1007/978-3-662-53622-3.

M. J. Garrido-Atienza, P. E. Kloeden, and A. Neuenkirch. Discretization of stationary solutions of stochastic systems driven by fractional Brownian motion. *Applied Mathematics & Optimization*, 60 (2):151–172, 2009. doi: 10.1007/s00245-008-9062-9.

M. Hairer. Ergodicity of stochastic differential equations driven by fractional Brownian motion. *The Annals of Probability*, 33(2):703–758, 2005. doi: 10.1214/009117904000000892.

M. L. Kleptsyna and A. Le Breton. Statistical analysis of the fractional Ornstein–Uhlenbeck type process. *Statistical Inference for Stochastic Processes*, 5(3):229–248, 2002. doi: 10.1023/A:1021220818545.

S. Kusuoka and D. W. Stroock. The partial Malliavin calculus and its application to non-linear filtering. *Stochastics*, 12(2):83–142, 1984. doi: 10.1080/17442508408833296.

X.-M. Li and J. Sieber. Slow-fast systems with fractional environment and dynamics. *The Annals of Applied Probability*, 32(5):3964–4003, 2022. doi: 10.1214/22-AAP1779.

X.-M. Li, F. Panloup, and J. Sieber. On the (non)stationary density of fractional-driven stochastic differential equations. *The Annals of Probability*, 51(6):2056–2085, 2023. doi: 10.1214/23-AOP1638.

Y. Liu, E. Nualart, and S. Tindel. LAN property for stochastic differential equations with additive fractional noise and continuous time observation. *Stochastic Processes and their Applications*, 129(8): 2880–2902, 2019. doi: 10.1016/j.spa.2018.08.008.

A. Neuenkirch and S. Tindel. A least square-type procedure for parameter estimation in stochastic differential equations with additive fractional noise. *Statistical Inference for Stochastic Processes*, 17 (1):99–120, 2014. doi: 10.1007/s11203-013-9084-z.

I. Nourdin and G. Peccati. Cumulants on the wiener space. *Journal of Functional Analysis*, 258(11): 3775–3791, 2010. doi: 10.1016/j.jfa.2010.02.003.

I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*, volume 192 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012. doi: 10.1017/CBO9781139084659.

D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and Its Applications. Springer, Berlin, Heidelberg, second edition, 2006. doi: 10.1007/3-540-28329-3.

D. Nualart and M. Zakai. The partial Malliavin calculus. In J. Azéma, M. Yor, and P. A. Meyer, editors, *Séminaire de Probabilités XXIII*, volume 1372 of *Lecture Notes in Mathematics*, pages 362–381. Springer, Berlin, Heidelberg, 1989. doi: 10.1007/BFb0083986.

V. Pipiras and M. S. Taqqu. Integration questions related to fractional Brownian motion. *Probability Theory and Related Fields*, 118(2):251–291, 2000. doi: 10.1007/s440-000-8016-7.

M. Reed and B. Simon. *Methods of Modern Mathematical Physics: I. Functional Analysis*. Academic Press, New York, revised and enlarged edition, 1980.

S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishers, 1993.